Derivative Pricing Based on the Exchange Rate in a Target Zone with Realignment∗

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Abstract

We propose a tractable model for the exchange rate in a target zone with realignment. The target zone exchange rate dynamics is assumed to obey a bounded regular diffusion with two-sided unattainable barriers. The realignment is modeled as a continuous-time two-state Markov chain. Under the stationary setting of the Markov chain, a general pricing formula for the derivative written on the exchange rate is derived in the presence of the realignment risk. The Jacobi diffusion model is studied as an example and numerical results are presented for illustration.

Keywords: Target zone exchange rate; currency derivative pricing; bounded diffusion; Markov chain; realignment; Jacobi diffusion.

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JEL classification: C13; C15; F31
1 Introduction

The purpose of this article is to present a tractable model for the exchange rate in a target zone with realignment risk and derive an explicit pricing formula for the derivatives written on the exchange rate. A well documented example of target zones may be the European Monetary System (EMS), which was implemented from 1979 through January 1, 1999, when the euro was introduced. After that, some European currencies began to carry out (formally or informally) target zone with the euro.

In the target zone regime, the exchange rate has some pre-specified lower and upper boundaries. The central banks should perform interventions in the foreign exchange markets in order to maintain the boundaries. This characteristic (boundedness) stimulates the researchers to use bounded diffusions to model the dynamics of the exchange rate in a target zone (see, e.g., Bertola and Caballero [2], Bertola and Svensson [3], De Jong et al. [4], Krugman [7], Larsen and Sørensen [8] and Svensson [13, 14]).

Another characteristic of the target zone is that it may not be fully credible, that is, most exchange rate target zones have experienced realignments (or adjustments). This phenomenon has been considered by many scholars. Among others, Bertola and Caballero [2] supposed the realignments occur at the edges of the exchange rate fluctuation. De Jong et al. [4] proposed a target zone model by employing a modified two-limit version of the CIR model, and modeled the parity adjustments by a Poisson process. Yu [18] explored the realignment risk of the Chinese Yuan, where the upward (appreciates) and downward (depreciates) realignment intensities were parameterized as exponential jump diffusions.

In this paper, in order to capture both of the boundedness of the target zone exchange rate and the realignment risk of the target zone, we model the dynamics of the exchange rate in a target zone by a bounded Markov regular diffusion with two-sided unattainable barriers, while the realignment is modeled by a continuous-time two-state Markov chain. The two states denote upward (appreciates) and downward (depreciates) realignments for the exchange rate, respectively. The realignment rule follows a similar manner as that in Svensson [13] (see Section 2 for details). Then using the developed model and under the stationary setting of the Markov chain, we derive a general pricing formula for the derivatives written on the target zone exchange rate in the presence of the realignment risk. A closed-form solution is obtained in the case that the exchange rate in a target zone is modeled as a Jacobi diffusion. Furthermore a numerical experiment is presented to illustrate the effect of the realignments.
The structure of the paper is as follows: Section 2 presents the mathematical description of the model. Section 3 devotes to deriving the pricing formula for an European-style derivative written on the exchange rate in a target zone with realignment risk. A closed-form expression is obtained when the exchange rate in a target zone is modeled by a so-called Jacobi diffusion. In Section 4, we provide some numerical experiments to illustrate the effect of the realignment risk. Section 5 concludes the paper and gives some comments related to barrier-style payoff functionals.

2 The model description

In this section, we shall present the mathematical description of the target zone exchange rate model with realignment risk.

Let \((W_t)_{t \geq 0}\) be a standard Brownian motion on the physical probability space \(\Lambda = (\Omega, \mathcal{F}, P)\). Suppose the spot exchange rate\(^1\) process \((S_t)_{t \geq 0}\) without the realignment follows a bounded Markov regular diffusion process\(^2\) in the target zone \(I = (a, b)\) with \(a, b \in \mathbb{R}\) and \(a < b\). We further assume that the Markov infinitesimal generator \(A\) of the bounded diffusion process admits the form:

\[
Af(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad x \in I,
\]

(2.1)

with the domain of the definition

\[
D(A) = \{ f \in C^2_b(I); Af \in C_b(I), \text{ with the appropriate boundary conditions at } \partial I \},
\]

where the coefficients \(\mu(x)\) and \(\sigma(x)\) in (2.1) are assumed to be regular enough such that the Markov infinitesimal generator \(A\) is well defined and unique.

Due to the unattainability of the boundary \(\partial I\) of the target zone, the boundaries \(\partial I\) are entrance or natural in light of the boundary classification for the regular diffusion in Ch.15 of Karlin and Taylor [6].

It’s known that if the boundary is natural, no boundary condition is needed. However, if the boundary is entrance, the boundary conditions at \(\partial I\) can be written

\(^1\)We refer to the logarithm of the true exchange rate as the exchange rate in this paper.

\(^2\)In [2, 13], the authors used the regulated Brownian motion to established the fundamental determinants of the exchange rate.
in the form:

\[
\lim_{x \downarrow a} \frac{f'(x)}{s(x)} = \lim_{x \uparrow b} \frac{f'(x)}{s(x)} = 0, \tag{2.2}
\]

where the scale density (see e.g. Karlin and Taylor [6]) is given by

\[
s(x) = \exp \left( -\int_{x}^{x} \frac{\mu(y)}{\sigma^2(x)} dy \right), \quad \text{for } x \in I. \tag{2.3}
\]

The scale density \( s(x) \) can be defined up to a scaling constant \( c \), i.e., \( s(x) \rightarrow cs(x) \) (see, e.g., [6]).

On the other hand, we suppose the physical probability space \( \Lambda \) also supports a upward or downward realignment \((R_t)_{t \geq 0}\) process, which is modeled as a continuous-time two-state Markov chain

\[
dR_t = (U - D) \left[ \gamma_D(R_{t-})dN_t^D + \gamma_U(R_{t-})dN_t^U \right], \tag{2.4}
\]

where \( 0 < D < 1 < U \) denote the realignment amplitudes, \( N_t^U \) and \( N_t^D \) are two independent Poisson processes with a common state-dependent intensity \( \lambda_t = \lambda(R_t) \). Moreover \( N_t^U \) and \( N_t^D \) are assumed to be independent of the spot exchange rate \((S_t)_{t \geq 0}\). To describe the two states of the realignment with \( R_t \in \{U, D\} \) for all \( t \geq 0 \), the following relations are necessary

\[
\begin{align*}
\gamma_D(D) &= -\gamma_U(U) = 1, \\
\gamma_D(U) &= \gamma_U(D) = 0.
\end{align*}
\]

In view of (2.4), the associated \( Q \)-matrix and transition matrix \( P(t) \) of the realignment \((R_t)_{t \geq 0}\) are respectively given by

\[
Q = \begin{pmatrix}
-\lambda(D) & \lambda(D) \\
\lambda(U) & -\lambda(U)
\end{pmatrix}, \tag{2.5}
\]

and

\[
P(t) = \frac{1}{\lambda} \begin{pmatrix}
\lambda(U) + \lambda(D)e^{-[\lambda(D) + \lambda(U)]t} & \lambda(D) - \lambda(D)e^{-[\lambda(D) + \lambda(U)]t} \\
\lambda(U) - \lambda(U)e^{-[\lambda(D) + \lambda(U)]t} & \lambda(D) + \lambda(U)e^{-[\lambda(D) + \lambda(U)]t}
\end{pmatrix}, \tag{2.6}
\]

with \( \lambda = \lambda(D) + \lambda(U) \).

In this paper, we adopt the realignment manner in Svensson [13], that is, the exchange rate is formulated by \( X_t = S_t R_t \) for each \( t \geq 0 \).
3 Pricing of the derivatives

In this section, we derive the pricing formula for the derivative written on the target zone exchange rate with realignment \((X_t)_{t \geq 0}\) defined in Section 2. As an example, we will model the dynamics of exchange rates in a target zone by the so-called Jacobi diffusion.

For parsimony, we suppose the market price of the volatility risk \(\alpha_t \equiv 0\). Furthermore, let \(\eta_t = \eta(R_t)\) be the realignment risk premium due to the existence of the realignment \(R_t\) at date \(t\). Thus we can identify the equivalent martingale measure \(Q\) by defining \(\frac{dQ}{dP}|_{\mathcal{F}_t} = \ell_t\), where \(\ell_t\) satisfies \(E_P[\ell_t] = 1\) and

\[
\frac{d\ell_t}{\ell_{t-}} = (\eta_t - 1)dZ_t, \tag{3.1}
\]

in which \((Z_t)_{t \geq 0}\) is defined by (A.1) in the Appendix. As a consequence, the Brownian motion \((W_t)_{t \geq 0}\) under \(P\) is also a Brownian motion under \(Q\). While

\[
dZ_t^Q := dR_t - (U - D)\eta_t\lambda(R_t)\gamma R_t(R_t)dt \tag{3.2}
\]

is a martingale under \(Q\). That is to say, the compensator of \(dR_t\) becomes \((U - D)\eta_t\lambda(R_t)\gamma R_t(R_t)dt\) under the risk-neutral measure \(Q\).

3.1 Implications of the realignment

In this subsection, we consider the implications of incorporating the realignment risk into the target zone exchange rate dynamics. Since the realignment \((R_t)_{t \geq 0}\) is formulated as a two-state Markov chain, the realignment spot exchange rate \(X_t = S_t R_t\) produces the jump phenomena.

The following lemma identifies the infinitesimal generator of the spot exchange rate \((X_t)_{t \geq 0}\) with the realignment \((R_t)_{t \geq 0}\) in the different realignment states.

Lemma 3.1. Let \(g_i(x) = f(ix)\) with \(i \in \{U, D\}\) and \(f(i \cdot) \in D(A)\). Define the infinitesimal generator

\[
\mathcal{B}g_i(x) = \lim_{t \downarrow 0} \frac{E_{x,i} [f(X_t)]}{t}.
\]

Then

\[
\mathcal{B}g_D(x) = \mu(x)g'_D(x) + \frac{1}{2} \sigma^2(x)g''_D(x) + \lambda(D) [g_U(x) - g_D(x)], \tag{3.3}
\]

\[
\mathcal{B}g_U(x) = \mu(x)g'_U(x) + \frac{1}{2} \sigma^2(x)g''_U(x) + \lambda(U) [g_D(x) - g_U(x)]. \tag{3.4}
\]
The equations (3.3) and (3.4) are useful in the pricing of various derivatives concerning the realignment target zone exchange rate \((X_t)_{t \geq 0}\), which shall be discussed in the following subsections.

3.2 The pricing formula

In this subsection, we derive a pricing formula for the derivative written on the target zone exchange rate \((X_t)_{t \geq 0}\).

Let \(c(t, x)\) be the cash flow rate paid by the security per unit of time and \(g(x)\) the payoff of the derivative security at maturity, which are all dependent on the realignment exchange rate level \(x\). Then the pricing formula follows the form (see, e.g., Aït-Sahalia [1]):

\[
P(t, x) = \mathbb{E}^Q \left[ g(X_T) e^{-rT} + \int_{T-t}^T e^{-r(s+t-T)} c(s, X_s) ds \ \bigg| S_{T-t} = x \right],
\]

where \(r = r^D - r^F\) denotes the domestic/foreign interest differential and is assumed to be known and constant. It is not difficult to show that

\[
P(t, x) = P_D(t, x) Q(R_{T-t} = D) + P_U(t, x) Q(R_{T-t} = U),
\]

(3.5)

where the realignment state prices \(P_i(t, x) (i \in \{U, D\})\) admit the following form:

\[
P_i(t, x) = \mathbb{E}^Q \left[ g(X_T) e^{-rT} + \int_{T-t}^T e^{-r(s+t-T)} c(s, X_s) ds \ \bigg| S_{T-t} = x, R_{T-t} = i \right].
\]

(3.6)

To obtain the explicit expression of the price \(P(t, x)\) in (3.5), we have to calculate the probabilities \(q_i(t) = Q(R_t = i)\) with \(i \in \{U, D\}\). Although these probabilities are easy to be calculated by using the transition matrix, we will adopt a much simpler setting: we presume the stationarity of the realignment process \((R_t)_{t \geq 0}\) with the stationary distribution \(\pi = (\pi_U, \pi_D)\) under the risk-neutral measure \(Q\). Thus \(q_i(T - t) \equiv \pi_i\) with \(i \in \{U, D\}\). It should be noted that the element \(\lambda(i)\) in the \(Q\)-matrix and transition matrix \(P(t)\) shall be replaced by \(\eta(i)\lambda(i)\) with \(i \in \{U, D\}\) under the risk-neutral measure \(Q\). Therefore, we have

\[
\pi_D = \frac{\eta(U)\lambda(U)}{\eta(D)\lambda(D) + \eta(U)\lambda(U)}, \quad \pi_U = \frac{\eta(D)\lambda(D)}{\eta(D)\lambda(D) + \eta(U)\lambda(U)}.
\]

(3.7)

Thus the only remaining issue is to derive the realignment state prices \(P_i(t, x)\) with \(i \in \{U, D\}\). Toward this end, we have
Theorem 3.1. Suppose the boundaries $\partial I = \{a, b\}$ are entrance (resp. natural). Then the realignment state price $P_i(t, x)$ with $i \in \{U, D\}$ admits the form:

$$P_D(t, x) = \frac{L(t, x) + \bar{\lambda}_D K(t, x)}{\lambda_D + \lambda_U},$$  \hspace{1cm} (3.8)$$

$$P_U(t, x) = \frac{L(t, x) - \bar{\lambda}_U K(t, x)}{\lambda_D - \lambda_U},$$

where $\bar{\lambda}_D = \eta(D)\lambda(D)$ and $\bar{\lambda}_U = \eta(U)\lambda(U)$ denote the realignment intensities. In addition, $K(t, x)$ and $L(t, x)$ satisfy the following PDEs respectively

$$\begin{cases}
K_t(t, x) = \mu(x)K_x(t, x) + \frac{1}{2}\sigma^2(x)K_{xx}(t, x) \\
\quad + [\bar{\lambda}_D + \bar{\lambda}_U - r] K(t, x) + [c(t, Dx) - c(t, Ux)], \\
\lim_{x \downarrow a} K_x(t, x) = \lim_{x \uparrow b} K_x(t, x) = 0, \\
K(0, x) = g(Dx) - g(Ux),
\end{cases}$$  \hspace{1cm} (3.9)$$

$$\begin{cases}
L_t(t, x) = \mu(x)L_x(t, x) + \frac{1}{2}\sigma^2(x)L_{xx}(t, x) \\
\quad - rL(t, x) + [\bar{\lambda}_D(t, x) + \bar{\lambda}_U(t, x)], \\
\lim_{x \downarrow a} L_x(t, x) = \lim_{x \uparrow b} L_x(t, x) = 0, \\
L(0, x) = \bar{\lambda}_Dg(Ux) + \bar{\lambda}_Ug(Dx),
\end{cases}$$  \hspace{1cm} (3.10)$$

where $\bar{\lambda}_i(t, x) = \bar{\lambda}_i c(t, ix)$ with $i \in \{D, U\}$ and the subscripts in $K$ and $L$ denote the derivatives with respect to the associated variables.

3.3 An example: The Jacobi diffusion model

In the subsection, we adopt a Jacobi diffusion to model the target zone spot exchange rate without the realignments. The Jacobi diffusion has the drift

$$\mu(x) = \alpha(\theta - x),$$

and the volatility

$$\sigma(x) = \sigma\sqrt{(x + \beta)(\beta - x)},$$
for all $x \in I = (-\beta, \beta)$ with $\alpha > 0$, $\beta > 0$, $\sigma > 0$ and $\theta \in I$. Let’s define

$$c_1 = \frac{\alpha(\beta + \theta)}{\sigma^2 \beta}, \quad c_2 = \frac{\alpha(\beta - \theta)}{\sigma^2 \beta}.$$ 

**Assumption 1.** Assume that the Feller conditions $c_1 \geq 1$ and $c_2 \geq 1$ hold.

It is not difficult to show that, under Assumption 1, the boundaries $\partial I$ are (unattainable) entrance and the Jacobi diffusion is ergodic, which guarantees the existence of the invariant distribution (a shifted and re-scaled Beta distribution on $(-\beta, \beta)$). Intuitively, the economic interpretation of the Feller conditions is that the regulation parameter $\alpha$ of the central banks should be strong enough compared with the market volatility $\sigma$ to keep the process stationary (see also De Jong et al. [4] and Larsen and Sørensen [8]).

In what follows, we start to seek a solution for $P_D(t,x)$ and $P_U(t,x)$ in (3.6). Let the coupon $c(t,q) \equiv c$ be a positive constant. In light of Theorem 3.1, Equations (3.9) and (3.10) are reduced to

$$\begin{cases} 
K_t(t,x) = \alpha(\theta - x)K_x(t,x) + \frac{\sigma^2}{2}(x + \beta)(\beta - x)K_{xx}(t,x) \\
+ \left[\bar{\lambda}_D + \bar{\lambda}_U - r\right]K(t,x), \\
K(0,x) = g(Dx) - g(Ux), \\
\lim_{x \uparrow 0} \frac{K_x(t,x)}{s(x)} = \lim_{x \uparrow b} \frac{K_x(t,x)}{s(x)} = 0,
\end{cases} \quad (3.11)$$

and

$$\begin{cases} 
L_t(t,x) = \alpha(\theta - x)L_x(t,x) + \frac{\sigma^2}{2}(x + \beta)(\beta - x)L_{xx}(t,x) \\
- rL(t,x) + \left[\bar{\lambda}_D + \bar{\lambda}_U\right]c, \\
L(0,x) = \bar{\lambda}_D g(Ux) + \bar{\lambda}_U g(Dx), \\
\lim_{x \uparrow 0} \frac{L_x(t,x)}{s(x)} = \lim_{x \uparrow b} \frac{L_x(t,x)}{s(x)} = 0.
\end{cases} \quad (3.12)$$

Observe Equations (3.11) and (3.12). Let

$$\hat{L}(t,x) = L(t,x) - \frac{\left[\bar{\lambda}_D + \bar{\lambda}_U\right]c}{r},$$

then $\hat{L}(t,x)$ satisfies a similar equation as (3.11). Consequently we only need to find the solution to Equation (3.11) and we try to look for the solution which admits the following form:

$$K(t,x) = h(t) \phi(x).$$

8
From Equation (3.11), it follows that
\[
\frac{h'(t)}{h(t)} = \frac{\alpha(\theta - x)\phi'(x) + \frac{\sigma^2}{2}(x + \beta)(\beta - x)\phi''(x) + \bar{r}\phi(x)}{\phi(x)} = -\rho,
\]
where \(\bar{r} := \bar{\lambda}_D + \bar{\lambda}_U - r\) and \(\rho \geq 0\). Thus \(\phi(x)\) satisfies that
\[
\frac{\sigma^2}{2}(x + \beta)(\beta - x)\phi''(x) + \alpha(\theta - x)\phi'(x) + (\bar{r} + \rho)\phi(x) = 0. \tag{3.13}
\]
In fact, second-order ODE (3.14) is equivalent to a Sturm-Liouville equation. Since the boundaries \(\partial I\) are entrance, there are infinite many pairs \(\{\rho_i, \phi_i(x)\}_{i=1}^{\infty}\) with \(\rho_i \to \infty\) and \(\phi_i(x) \neq 0, \ i \geq 1\) such that (3.14) holds. It follows that
\[
K(t, x) = \sum_{i=1}^{\infty} C_i e^{-\rho_i t} \phi_i(x), \tag{3.14}
\]
where \(C_i, \ i \geq 1\) are undetermined constants which can be determined by the initial condition. As for the Sturm-Liouville boundary value problem and the corresponding numerical implementation, refer to Zettl [19] and Pryce [10].

4 The numerical experiment

In this section, we perform a numerical experiment on pricing the European currency option.

Consider the following Jacobi diffusion for modeling the target zone exchange rate without the realignments,
\[
dS_t = -\alpha S_t dt + \sigma \sqrt{(S_t + \beta)(\beta - S_t)} \ dW_t. \tag{4.1}
\]

We intent to price a European call option written on the realignment exchange rates \((X_t)_{t \geq 0}\). Throughout this section, we adopt the following preference parameters.

<table>
<thead>
<tr>
<th>Table 1: The preference parameters.</th>
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<tbody>
<tr>
<td>The annual interest rate (r^D)</td>
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<tr>
<td>The annual foreign interest rate (r^F)</td>
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<tr>
<td>The appreciation rate (\alpha)</td>
</tr>
<tr>
<td>The maximum deviation (\beta)</td>
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<tr>
<td>The volatility (\sigma^2)</td>
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<tr>
<td>The initial FX rate (S_0)</td>
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<tr>
<td>The realignment risk premiums</td>
</tr>
</tbody>
</table>
Suppose the realignment process \((R_t)_{t \geq 0}\) is stationary under \(Q\) with stationary distribution \(\pi = (\pi_D, \pi_U)\). Then we have

\[
P(t, x) = \pi_D \mathbb{E}_Q [(DS_T - K)^+]|S_{T-t} = x] + \pi_U \mathbb{E}_Q [(US_T - K)^+]|S_{T-t} = x]
\]

\[
= \pi_D \int_{-K/D}^{\beta} (Dy - K)p(t, x, y)dy + \pi_U \int_{-K/U}^{\beta} (Uy - K)p(t, x, y)dy,
\]

\[ (4.2) \]

where \(p(t, x, y)\) denotes the transition density of the Jacobi diffusion process (see p. 335 in Karlin and Taylor [6]).

We first consider the case: \(U = D = 1\), i.e., there is no realignment risk. Figure 1 depicts the prices of the European call options against the maturity with strike prices equal to \(-0.1, 0\) and \(0.1\), respectively. This plot reflects the typical styles of the call option price: increasing with maturity and decreasing with strike price.

![Figure 1: The prices of the European call options against the maturities \(T = 1/4, 1/2, 1, 2\). The solid, dashed and dotted lines correspond to strike prices \(K = 0.1, 0\) and \(-0.1\), respectively.](image1)

We then turn to the cases with realignment risk. Assume that \(D = 0.8, U = 1.2, K = 0\) and \(\eta(D) = \eta(U) = 1.1\). Figure 2 depicts the results. The solid line is the un-realignment \((D = U = 1)\) prices. The dashed line corresponds to the case
\( \lambda(D) = 1, \lambda(U) = 2 \). The dotted line corresponds to the case \( \lambda(D) = 2, \lambda(U) = 1 \). The dot dashed line above corresponds to the case \( \lambda(D) = 1, \lambda(U) = 0 \). The dot dashed line below corresponds to the case \( \lambda(D) = 0, \lambda(U) = 1 \). From Figure 2, we find that the appearance of the realignment \((R_t)_{t \geq 0}\) has a notable effect on the option prices. Specifically, if the ratio of the intensities \( \frac{\lambda(D)}{\lambda(U)} \) is large (which means that the likelihood of the Markov chain being the appreciate state is great), the (call) option prices will be high. And this phenomenon becomes more apparent as the time-to-maturity increasing.

5 Concluding remarks

In this paper, we have proposed a reasonable and tractable model for the exchange rate in a target zone with realignment. The dynamics of the exchange rate in target zone was modeled by a bounded regular diffusion with two-sided unattainable barriers, while the realignment was realized by a two-state Markov chain. Then we studied the pricing of some currency derivatives with an European-style pay-off functional and provided a numerical experiment in which the pre-realignment exchange rate was modeled by the so-called Jacobi diffusion.

In general the method developed in this paper can not be applied to the pricing of barrier options, which has the following payoff functional (take the down-and-out option as an example)

\[
g(X_T) \mathbf{1}_{\{\tau > T\}}, \quad \text{with} \quad \tau := \inf \{t \geq 0 : X_t \leq d(t)\},
\]

where \( d(t) \in I \) is some barrier function. However, if the barrier function takes the special form \( d(t) = d \times R_t \) with a constant \( d \) (which means that the barrier is also adjusted by the realignment process, and now \( \tau = \inf \{t \geq 0 : S_t \leq d\} \)), we can apply our approach to the pricing of the barrier option by replacing the boundary condition (in (3.9)-(3.12)) at the lower boundary \( a \) by the homogeneous Dirichlet boundary condition at the barrier \( d \) (since now the barrier \( d \) amounts to a killing boundary). The expansion formula (3.14) can also be derived but with different \( \{p_i, \phi_i(x)\}_{i=1}^{\infty} \). Moreover, the pricing formula for the above payoff functional goes as follows (see also (4.2)):

\[
\hat{P}(t, x) = \pi_D \int_d^\beta g(Dy)\hat{p}(t, x, y)dy + \pi_U \int_d^\beta g(Uy)\hat{p}(t, x, y)dy,
\]

where \( \hat{p} \) is the transition density of the process \( S \) killed at the first hitting time of the level \( d \). As for the killed Jacobi diffusion, its transition density can be derived
in terms of spectral expansion (the eigenvalues and the eigenfunctions have been found in [16]).

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Appendix: Proofs

Proof of Lemma 3.1. The proof of Lemma 3.1 is based on the fact that the compensator of \((R_t)_{t\geq 0}\) equals \((U - D)\lambda(R_t)\gamma(R_t)\). That is, the process

\[
\begin{align*}
\text{d}Z_t &= \text{d}R_t - (U - D)\lambda(R_t)\gamma(R_t)\text{d}t \\
&= (U - D)\lambda(R_t)\gamma(R_t)\text{d}t
\end{align*}
\]  

(A.1)
is a \((\mathbb{P}, \mathcal{F}_t)\)-martingale. Since the realignment spot exchange rate \(X_t = S_tR_t\), we use the integration by parts formula to conclude that

\[
\begin{align*}
\text{d}X_t &= R_t\text{d}S_t + S_t\text{d}R_t \\
&= R_t\mu(S_t)\text{d}t + R_t\sigma(S_t)\text{d}W_t + S_t\text{d}R_t.
\end{align*}
\]  

(A.2)

Therefore, it follows from the Itô rule with jumps and (A.2) that

\[
\begin{align*}
\text{d}f(X_t) &= f'(X_t)R_t\mu(S_t)\text{d}t + f'(X_t)R_t\sigma(S_t)\text{d}W_t + \frac{1}{2}f''(X_t)R_t^2\sigma^2(S_t)\text{d}t \\
&\quad + \left[f(X_t) + (U - D)S_t\gamma(R_t - \gamma(R_t))\right]\left[\lambda(R_t)\text{d}N_t - \lambda(R_t)\text{d}t\right] \\
&\quad + \lambda(R_t)\left[f(X_t) + (U - D)S_t\gamma(R_t)\right] - f(X_t)\text{d}t.
\end{align*}
\]  

Then the conclusion follows.

\[\Box\]

Proof of Theorem 3.1. Using Lemma 3.1, applying the Feynman-Kac formula to (3.6), it follows that \(P_D(t, x)\) and \(P_U(t, x)\) solve the following equations respectively

\[
\begin{align*}
\frac{\partial P_D(t, x)}{\partial t} &= \mu(x)\frac{\partial P_D(t, x)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 P_D(t, x)}{\partial x^2} + \lambda_D[P_D(t, x) - P_D(t, x)] \\
&\quad - rP_D(t, x) + c(t, Dx), \\
\frac{\partial P_U(t, x)}{\partial t} &= \mu(x)\frac{\partial P_U(t, x)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 P_U(t, x)}{\partial x^2} + \lambda_U[P_D(t, x) - P_U(t, x)] \\
&\quad - rP_U(t, x) + c(t, Ux).
\end{align*}
\]

Then the desired result is concluded by defining

\[
K(t, x) = P_D(t, x) - P_U(t, x), \quad \text{and} \quad L(t, x) = \lambda_D P_U(t, x) + \lambda_U P_D(t, x).
\]
References


