The forward interest rate curve driven by a Lévy random field with a kernel-correlated Gaussian component

Lijun Bo¹, Yongjin Wang², Xuewei Yang³∗

¹Department of Mathematics, Xidian University, Xi’an 710071, China
e-mail: bolijunnk@gmail.com

²School of Business, Nankai University, Tianjin 300071, China,
and Department of Statistics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, U.K.
e-mail: yjwang@nankai.edu.cn

³School of Mathematical Sciences and TEDA Institute of Computational Finance, Nankai University, Tianjin 300071, China, and
Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
e-mail: xuyangnk@yahoo.com.cn

First version: January 30, 2010 This version: October 18, 2010

Abstract: In this paper, we propose a term structure of forward rates driven by a Lévy random field under the HJM framework. The Lévy random field is composed of a kernel-correlated Gaussian random field and a Poisson random field. We will give a criterion to preclude arbitrage under the risk-neutral measure. Finally, an interest rate derivative is priced under this framework.

Keywords and phrases: Forward interest rate, Lévy random field, HJM model, derivative pricing.

AMS 2000 subject classifications: 60H30, 49J22, 90A09.

JEL classifications: E43.

1. Introduction

In the last decades, the term structure of interest rate has been playing an important role in the financial markets, and so there have been extensive studies in the literature (see, e.g., Filipović [16, 17] and the references therein). Some well-known finite-dimensional term-structure models include the CIR model in Cox, Ingersoll and Ross [7], the HJM model in Heath, Jarrow, Morton [22], the Ho-Lee model in [23], the Hull-White model in [24], the Vasicek model in [41], etc. Recently, in [6], Bjork and Svensson studied the existence of the finite dimensional realizations for the nonlinear forward rate models. Filipović and Teichmann [18] provided the characterization for the finite-dimensional HJM models that admit arbitrary initial yield curves. On the other hand, an infinite-dimensional term-structure model for forward rates was initiated by Musiela [32], which was successively developed by Aihara and Bagchi [1], Cont [8], Santa-Clara and Sornette [37], Sornette [40], etc. Most recently, Marinelli [31] studied the local well-posedness of the Musiela’s SPDE with Lévy noise. In the above mentioned

∗Corresponding author.
infinite-dimensional models, the forward rates were described as some stochastic processes in some specific curves spaces. Typically, those stochastic processes were described in terms of the solutions to some first-order hyperbolic stochastic equations or of some second-order stochastic heat equations.

In this paper we shall describe the term structure of interest rates by the so-called random field (or stochastic string) models for the forward interest rate. This type of term-structure models were originated by Kennedy [28] (see also Kennedy [29]), in which the author concentrated on the Gaussian random field and obtained a drift condition for the forward rate process to preclude arbitrage under the risk-neutral measure. Goldstein [21] and Kimmel [30] generalized the Kennedy model by introducing the conditional volatility, which have resulted in non-Gaussian random fields. As far as we know, the random field models have several advantages comparing to the other type of models (see, e.g., [21, 30]), one of which is that the random field models offer a parsimonious description of term structure dynamics while eliminating the self-inconsistent practice of re-calibration.

As it was documented in Björk et al. [4, 5], the empirical evidence shows that the interest rate models should incorporate the jump risk (see also [20, 27, 38]). Thus the recent works usually took the occurrence of jumps into account. Among others, Eberlein et al. [12–15] and Jakubowski and Zabczyk [26] investigated the term structure models driven by Lévy processes. Filipovic and Tappe [19] explored the existence of the Lévy term structure models. In Özkan and Schmidt [35], the forward rate curve was driven by an infinite-dimensional Lévy process, which took values on a Hilbert-space. Albeverio, Lytvynov and Mahnig [2] modeled the forward interest rate by a Lévy field without a diffusive component. Most recently, Filipović, Tappe and Teichmann [20] studied the term structure models driven by Wiener processes and Poisson random measures.

In this paper, under the HJM framework, we build a tractable forward interest rates model driven by a Lévy random field, in which the continuous diffusive risk is formulated by a kernel-correlated Gaussian random field\(^1\), and the jump risk is described by a Poisson random field (which corresponds to a Poisson random measure, see Nunno [34] and Filipović, Tappe and Teichmann [20]). Our model is different from the existing Lévy driven forward interest rate models: firstly, comparing with the pure jump model in Albeverio, Lytvynov and Mahnig [2], we introduce a kernel-correlated Gaussian random field to the forward rate model; secondly, we use a Poisson random measure as in Filipovic, Tappe and Teichmann [20], while the continuous diffusive component is a kernel-correlated Gaussian random field (which is different from [20]) and the derivative pricing is also studied (which was not considered in [20]). Except for the advantages for the general random field models documented in [21] and [30], our model seems to have at least three additional advantages: (I) it can be made sufficiently flexible by choosing different kernel-functions; In particular, if we choose some specific kernel function, the Gaussian random field can be reduced to the classical Gaussian field, such as the Brownian sheet (see, e.g., Kennedy [28] and Walsh [42]); (II) the kernel-correlated structure for the Gaussian random field is tractable when we are concerned with the pricing of the interest rate derivatives with general payoffs (see Section 6 below); (III) compared with the random field models studied in [21, 28, 30], we introduce a jump component described by a Poisson random measure, which

\(^1\)The kernel-correlated Gaussian random field is well known in the field of stochastic partial differential equations (SPDEs). In fact its formal (or generalized) derivative is the so-called colored (or spatial-correlated) noise, which have been studied by many authors (see, e.g., Dalang [9], Mytnik, Perkins and Sturm [33] and Dalang and Mueller [10]).
is in consistent with the empirical evidence (see, e.g., [20]).

After presenting the detailed description of the forward rates model driven by the proposed Lévy field, a non-arbitrage condition (a generalized version of the drift condition presented in [22]) on the drift of the forward rates is achieved by using the Itô formula with jumps. Further, a risk-neutral probability measure is identified, and then the corresponding risk-neutral dynamics of the bond price, the forward interest rate and the instantaneous interest rate are derived respectively. Finally, based on the above mentioned risk-neutral measure, in spirit of Aihara and Bagchi [1] we derive the explicit pricing formula for an interest rate derivative with a general payoff.

The remainder of the paper is organized as follows. In the coming section, we begin with the definition of a Lévy random field and introduce the corresponding stochastic integral w.r.t. this Lévy field. In Section 3, we formulate the forward rate curves driven by the Lévy random field introduced in Section 2. Section 4 is devoted to deriving a criterion for precluding the arbitrage under the risk-neutral probability measure. Finally, in Section 5, we discuss and present the pricing of an interest rate derivative with a general payoff. Section 6 concludes.

2. Introduction of a Lévy random field and the stochastic integral

In this section, we introduce a Lévy random field, which is a combination of a kernel-correlated Gaussian random field and a Poisson random measure.

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space with \( \mathbb{P} \) being the physical (statistical) probability measure. Define

\[
Y^G := \{ Y^G(\phi); \ \phi \in D(\mathbb{R}^{d+1}) \}
\]

which is a mean-zero Gaussian process with the covariance functional

\[
J(\phi_1, \phi_2) = \mathbb{E} \left[ Y^G(\phi_1)Y^G(\phi_2) \right] = \int_0^\infty \int_{\mathbb{R}^{2d}} \phi_1(\xi_1, t)c(\xi_1 - \xi_2)\phi_2(\xi_2, t)d\xi_1d\xi_2dt,
\]

where the kernel function \( c: \mathbb{R}^d \to (0, \infty) \) is continuous and symmetric (i.e., \( c(-\xi) = c(\xi) \)). The typical examples of kernel functions are Dirac function and Riesz functions (see, e.g., Dalang [9]). Particularly, if the kernel function is the Dirac function, the Gaussian process can be reduced to the classical Gaussian field, such as the Brownian sheet (see, e.g., Walsh [42]).

As in Dalang [9] and Walsh [42], the above Gaussian process \( Y^G \) can be extended to a worthy martingale measure. In fact, for \( B \in B_b(\mathbb{R}^d) \), a bounded Borel subset of \( \mathbb{R}^d \), define

\[
M_t(B) = Y^G([0, t] \times B),
\]

and

\[
\mathcal{F}^0_t = \sigma(M_u(B), \ u \leq t, \ B \in B_b(\mathbb{R}^d)), \ \mathcal{F}^1_t = \mathcal{F}^0_t \vee \mathcal{N},
\]

\(^2 D(\mathbb{R}^{d+1}) \) denotes the topological vector space of functions \( \phi \in C^\infty_0(\mathbb{R}^{d+1}) \) with the topology corresponding to the following: \( \phi_n \to \phi \) if and only if: (i) there exists a compact set \( K \) of \( \mathbb{R}^{d+1} \) such that the support \( \text{supp}(\phi_n - \phi) \subset K \) for all \( n \), and (ii) \( D^\alpha \phi_n \to D^\alpha \phi \) uniformly on \( K \) for each multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \).
where $\mathcal{N}$ denotes the $\sigma$-field generated by all $\mathbb{P}$-null sets. Then

$$M = (M_t(B), \mathcal{F}_t^1, B \in \mathcal{B}_b(\mathbb{R}^d))$$

is a $\mathbb{P}$-martingale measure. Moreover it is a worthy measure with the following covariation measure

$$Q([0,t] \times A \times B) := \langle M(A), M(B) \rangle_t = t \int_{\mathbb{R}^2d} 1_A(\xi_1)c(\xi_1 - \xi_2)1_B(\xi_2)d\xi_1d\xi_2, \quad t > 0, \quad (2.1)$$

The dominating measure of the worthy measure $M$ is just $Q$. Note that for each $B \in \mathcal{B}_b(\mathbb{R}^d)$, the real-valued process $t \mapsto M_t(B)$ is a continuous $\mathbb{P}$-martingale, and

$$Y^G(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,\xi)Y^G(d\xi,dt), \quad \text{for} \quad \phi \in \mathcal{D}(\mathbb{R}^{d+1}).$$

In what follows, we define the stochastic integral with respect to the worthy martingale measure $Y^G$. We begin with a simple process $h(u,\xi,\omega) = 1_{(t_1,t_2]}(u)1_A(\xi)X(\omega)$, where $0 \leq t_1 < t_2 < \infty$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $X$ is an $\mathcal{F}_{t_1}$-measurable random variable. We define the stochastic integral $h \cdot Y^G$ by

$$(h \cdot Y^G)_t(B)(\omega) := \int_0^t \int_B h(u,\xi,\omega)Y^G(d\xi,du) = (M_{t \wedge t_2}(A \cap B) - M_{t \wedge t_1}(A \cap B))X(\omega).$$

Obviously, the stochastic integral $h \cdot Y^G$ is a $\mathbb{P}$-martingale measure. This definition can be extended to all finite linear combinations of such simple processes. Now let $h$ be a predictable random field, define

$$\|h\| := E \left[ \int_0^T \int_{\mathbb{R}^2d} |h(u,\xi_1)|c(\xi_1 - \xi_2)|h(u,\xi_2)|d\xi_1d\xi_2du \right],$$

and let $\mathcal{P}$ be the set of all predictable random fields $h$ with $\|h\| < \infty$. Then the space $(\mathcal{P}, \| \cdot \|)$ is complete, and for each element $h \in \mathcal{P},$

$$(u,\xi_1,\xi_2,\omega) \rightarrow h(u,\xi_1,\omega)h(u,\xi_2,\omega)$$

is integrable with respect to the measure $c(\xi_1 - \xi_2)d\xi_1d\xi_2du \mathbb{P}(d\omega)$. Moreover, it is easy to check that, for $h \in \mathcal{P},$

1. For each $B \in \mathcal{B}_b(\mathbb{R}^d)$, $t \mapsto (h \cdot Y^G)_t(B)$ is a continuous $(\mathbb{P}, (\mathcal{F}_t^1)_{t \geq 0})$-martingale;
2. The covariation process of $h \cdot Y^G$ is given by

$$Q_h([0,t] \times A \times B) := \langle (h \cdot Y^G)(A), (h \cdot Y^G)(B) \rangle_t$$

Preprint ver. file: levy_field-fr-r1-20101018.tex date: October 18, 2010
\[ = \int_0^t \int_A \int_B h(u, \xi_1)c(\xi_1 - \xi_2)h(u, \xi_2)d\xi_1d\xi_2du. \tag{2.3} \]

In this paper, we consider a Lévy random field \((Y(\xi, t))_{(\xi, t) \in \mathbb{R}^d \times (0, T]}\) with the following decomposition:

\[ Y(d\xi, dt) = Y^G(d\xi, dt) + Y^P(d\xi, dt), \tag{2.4} \]

where, for \((\xi, t) \in \mathbb{R}^d \times (0, T] \),

\[ Y^P(d\xi, dt) = J^P(d\xi, dt) - \nu(d\xi)dt \]

is a centered (compensated) Poisson random measure with the compensator \(\nu(d\xi)dt\) (here \(\nu(\cdot)\) is a Borel measure on \(\mathbb{R}^d\)), and it is independent of the kernel-correlated Gaussian field \(Y^G\). Further, we define

\[ \mathcal{F}_t^2 := \sigma(Y^P([0, u] \times B), u \leq t, B \in \mathcal{B}_b(\mathbb{R}^d)), \mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2, \]

and

\[ \Psi := \left\{ g(u, \xi, \omega); g \text{ is predictable and for each } t > 0, \int_0^t \int_{\mathbb{R}^d} \mathbb{E} \left[ |g(u, \xi)|^2 \right] \nu(d\xi)du < \infty \right\}. \tag{2.5} \]

Then for each random field \(g \in \Psi\), we can define the stochastic integral \(g \cdot Y^P\), and its covariation (random) measure is given by

\[ Q_g([0, t] \times A \times B) := \left[(g1_A \cdot Y^P), (g1_B \cdot Y^P)\right]_t = \int_0^t \int_{A \cap B} |g(u^-) \cdot \cdot|^2 J^P(d\xi, du). \tag{2.6} \]

For more details concerning the stochastic integral with respect to (w.r.t.) the Poisson random measure \(Y^P\), the interested readers may refer to Ikeda and Watanabe [25].

3. The forward rates driven by the Lévy random field

In this section, we describe the forward rate model driven by the Lévy random field in (2.4), and then derive the price dynamics of the zero-coupon bond.

Let \(T_f\) be a time horizon, and suppose that the bonds discussed below will mature at or before time \(T_f\). Let \(T > 0\) be the real maturity date of a zero-coupon default-free bond. For simplicity, we assume the face value of the bond is 1. Let \(P(t, T)\) be the time-\(t\) price of this bond with the maturity \(T \leq T_f\). Then the instantaneous default-free forward rates at time \(t\) for all maturities \(T \leq T_f\) are

\[ f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}. \tag{3.1} \]
Suppose that \((\mu(t, T - t))_{t \in [0, T]}\) is a random process, \((\sigma(\xi, t, T - t))_{(\xi, t) \in \mathbb{R}^d \times [0, T]} \in \mathcal{P}\), and \((\gamma(\xi, T - t))_{(\xi, t) \in \mathbb{R}^d \times [0, T]} \in \Psi\). Next we establish a model of the forward rate \(f(t, T)\) driven by the Lévy random field in (2.4) under the HJM framework. That is, the instantaneous forward rate in (3.1) admits the following dynamics

\[
d_t f(t, T) = \mu(t, T - t)dt + \int_{\mathbb{R}^d} \sigma(\xi, t, T - t)Y^G(d\xi, dt)
+ \int_{\mathbb{R}^d} \gamma(\xi, t, T - t)Y^P(d\xi, dt),
\]

which is equivalent to the following integral representation:

\[
f(t, T) = f(0, T) + \int_0^t \mu(u, T - u)du + \int_0^t \int_{\mathbb{R}^d} \sigma(\xi, u, T - u)Y^G(d\xi, du)
+ \int_0^t \int_{\mathbb{R}^d} \gamma(\xi, u - , T - u - )Y^P(d\xi, du),
\]

where \(\int_0^t \int_{\mathbb{R}^d} \sigma(\xi, u, T - u)Y^G(d\xi, du)\) is a stochastic integral with respect to the worthy martingale measure \(Y^G\), and \(\int_0^t \int_{\mathbb{R}^d} \gamma(\xi, u - , T - u - )Y^P(d\xi, du)\) is a stochastic integral with respect to the compensated Poisson random measure \(Y^P\) (see Section 2). Both of them are real-valued \((\mathbb{P}, (\mathcal{F}_t)_{t \geq 0})\)-martingales with mean-zero. Further, the covariance of the infinitesimal increment \(d_t f(t, T)\) can be given by

\[
\text{Cov}[d_t f(t, T), d_t f(t, T)] = \text{Cov}[\mu(t, T - t)dt, \mu(t, T - t)dt]
+ \int_{\mathbb{R}^d} \mathbb{E}[(\sigma(\xi_1, t, T - t)c(\xi_1 - \xi_2)\sigma(\xi_2, t, T - t))d\xi_1d\xi_2dt
+ \int_{\mathbb{R}^d} \mathbb{E}[|\gamma(\xi, t, T - t)|^2] \nu(d\xi)dt.
\]

On the other hand, the spot interest rate at time \(t\) is

\[r(t) = f(t, t),\]

which is the instantaneous rate that we lock at time \(t\) for borrowing at time \(t\). By letting \(T = t\) in (3.3), we obtain the dynamics of the spot interest rate as follows

\[
r(t) = f_0(t) + \int_0^t \mu(u, t - u)du + \int_0^t \int_{\mathbb{R}^d} \sigma(\xi, u, t - u)Y^G(d\xi, du)
+ \int_0^t \int_{\mathbb{R}^d} \gamma(\xi, u - , t - u - )Y^P(d\xi, du).
\]
By virtue of the relationship in (3.1), we get the non-annualized yield given by
\[ L(t, T) = -\log P(t, T) = \int_t^T f(t, y)dy, \]
which admits the following dynamics
\[
\begin{align*}
\frac{d_t L(t, T)}{L(t, T)} &= -f(t, t)dt + \int_t^T f(t, y)dy \\
&= (I_\mu(t, T-t) - r(t))dt + \int_{R^d} I_\sigma(\xi, t, T-t)Y_G(d\xi, dt) \\
&\quad + \int_{R^d} I_\gamma(\xi, t^-, T-t^-)Y_P(d\xi, dt), \text{ for } T > t,
\end{align*}
\]
where
\[
\begin{align*}
I_\mu(t, T-t) &= \int_t^T \mu(t, y-t)dy, \quad \text{and} \\
I_i(\xi, t, T-t) &= \int_t^T i(\xi, t, y-t)dy, \quad \text{for } i \in \{\sigma, \gamma\}.
\end{align*}
\]
Throughout this paper, we make the following assumptions on the jump measure and the kernel function:
\[
\nu(\cdot) \in \Lambda_\nu := \left\{ \text{Borel measure } \nu : \int_0^T \int_{R^d} E \left[ |I_\gamma(\xi, t, T-t)|^2 \right] \nu(d\xi)dt < \infty \right\},
\]
and
\[
c(\cdot) \in \Lambda_c := \left\{ \text{kernel function } c : \int_0^T \int_{R^d} E \left[ I_\sigma(\xi_1, t, T-t)c(\xi_1 - \xi_2)I_\sigma(\xi_2, t, T-t) \right] d\xi_1d\xi_2dt < \infty \right\}.
\]
Similarly as in (3.4), it holds that
\[
\text{Cov}[d_t L(t, T), d_t L(t, T)] = \text{Cov}[(I_\mu(t, T-t) - r(t))dt, (I_\mu(t, T-t) - r(t))dt]
\]
\[
+ \int_{R^d} \mathbb{E} [I_\sigma(\xi_1, t, T-t)c(\xi_1 - \xi_2)I_\sigma(\xi_2, t, T-t)] d\xi_1d\xi_2dt \\
+ \int_{R^d} \mathbb{E} [I_\gamma(\xi, t, T-t)^2] \nu(d\xi)dt.
\]
Applying Itô formula to \(\exp(-L(t, T))\), we conclude that
\[
\frac{d_t P(t, T)}{P(t, T)} = (r(t) - I_\mu(t, T-t))dt - \int_{R^d} I_\sigma(\xi, t, T-t)Y_G(d\xi, dt)
\]
\[
+ \frac{1}{2} \int_{R^d} I_\sigma(\xi_1, t, T-t)c(\xi_1 - \xi_2)I_\sigma(\xi_2, t, T-t)d\xi_1d\xi_2dt
\]
This provides a stochastic dynamics of the price for the zero-coupon bond with face value one.

4. The non-arbitrage criterion

In this section, we derive a non-arbitrage criterion for the forward rate model driven by the Lévy random field using the Itô formula with jumps.

Define the risk-neutral measure \( Q \) by

\[
\frac{dQ}{dP}igg|_{\mathcal{F}_t} = q(t),
\]

where the filtration \( \mathcal{F} \) is defined in Section 2, and the density process \( q(t) \) is given by (see, e.g., (1) and (8) in Bakshi, Carr and Wu [3] and Subsection 11.6.3 in Shreve [39])

\[
\frac{dq(t)}{q(t^-)} = \int_{\mathbb{R}^d} \psi(\xi, t) Y^G(d\xi, dt) + \int_{\mathbb{R}^d} (\varphi(\xi, t^-) - 1) Y^P(d\xi, dt)
\]

\[
q(0) = q(0^-) = 1,
\]

where \( \psi(\xi, t) \) denotes the market price of the diffusion risk corresponding to \( Y^G \) and \( \varphi(\xi, t) \) denotes the market price of the jump risk corresponding to \( Y^P \). The non-arbitrage assumption implies that the discounted bond price process is a martingale under the equivalent risk-neutral martingale measure. Recall the instantaneous interest rate \( r(t) \) given by (3.5). Now applying the Itô formula with jumps (see, e.g., Protter [36]) to the discounted bond price process

\[
(\hat{P}(t, T) := D(t)P(t, T))_{t \in [0, T]}
\]

with the discount factor

\[
D(t) := \exp\left(-\int_0^t r(u)du\right)
\]

and using the martingale property (under \( Q \)) of the discounted price process \( \hat{P}(t, T) \) yield the following arbitrage-free condition

\[
0 = -I_\mu(t, T - t) + \frac{1}{2} \int_{\mathbb{R}^{2d}} I_\sigma(\xi_1, t, T - t)c(\xi_1 - \xi_2)I_\sigma(\xi_2, t, T - t)d\xi_1 d\xi_2
\]

\[
+ \int_{\mathbb{R}^d} \psi(\xi, t)c(\xi_1 - \xi_2)I_\sigma(\xi_2, t, T - t)d\xi_1 d\xi_2
\]

\[
+ \int_{\mathbb{R}^d} (\varphi(\xi, t)e^{-I_\gamma(\xi, t, T-t)} - \varphi(\xi, t) + I_\gamma(\xi, t, T - t))\nu(d\xi).
\]
By differentiating both sides of the above equality w.r.t. the maturity $T$, we get

$$
\mu(t, T-t) = \int_{\mathbb{R}^d} \sigma(\xi_1, t, T-t)c(\xi_1 - \xi_2)I_\sigma(\xi_2, t, T-t)d\xi_1d\xi_2
+ \int_{\mathbb{R}^d} \psi(\xi_1, t)c(\xi_1 - \xi_2)\sigma(\xi_2, t, T-t)d\xi_1d\xi_2
+ \int_{\mathbb{R}^d} \gamma(\xi, t, T-t)(1 - \varphi(\xi, t)e^{-\gamma(\xi, t, T-t)})\nu(d\xi). \quad (4.5)
$$

The above equality can be viewed as the generalized version of the drift condition in Heath, Jarrow and Morton [22].

From now on, we will assume that the condition (4.5) is satisfied. Then, under this risk-neutral measure $\mathbb{Q}$, the dynamics of the bond price $(P(t, T))_{t \in [0, T]}$ is

$$
\frac{dP(t, T)}{P(t-, T)} = r(t)dt - \int_{\mathbb{R}^d} I_\sigma(\xi_2, t, T-t)M^G(d\xi, dt)
+ \int_{\mathbb{R}^d} (e^{-\gamma(\xi, t-, T-t-)} - 1)M^P(d\xi, dt). \quad (4.6)
$$

Substituting (4.5) into (3.3) yields the $\mathbb{Q}$-dynamics of the forward interest rate $(f(t, T))_{t \in [0, T]}$:

$$
f(t, T) = f_0(T) + \int_0^t \int_{\mathbb{R}^d} \sigma(\xi_1, u, T-u)c(\xi_1 - \xi_2)I_\sigma(\xi_2, u, T-u)d\xi_1d\xi_2du
+ \int_0^t \int_{\mathbb{R}^d} \sigma(\xi_2, u, T-u)M^G(d\xi, du) + \int_0^t \int_{\mathbb{R}^d} \gamma(\xi, u-, T-u-)M^P(d\xi, du)
+ \int_0^t \int_{\mathbb{R}^d} \gamma(\xi, u, T-u)(1 - e^{\gamma(\xi, u, T-u)})\varphi(\xi, u)\nu(d\xi)du, \quad (4.7)
$$

where the (random) $\mathbb{Q}$-martingale measures $M^G(d\xi, du)$ and $M^P(d\xi, du)$ are respectively defined by

$$
M^G(d\xi_2, du) = Y^G(d\xi_2, du) + \left(\int_{\mathbb{R}^d} \psi(\xi_1, u)c(\xi_1 - \xi_2)d\xi_1\right)d\xi_2du, \quad \text{and} \quad (4.8)
$$

$$
M^P(d\xi, du) = Y^P(d\xi, du) + (1 - \varphi(\xi, u))\nu(d\xi)du
= J^P(d\xi, du) - \varphi(\xi, u)\nu(d\xi)du. \quad (4.9)
$$

Again, by letting $T = t$ in (4.7), we get the $\mathbb{Q}$-dynamics of the instantaneous interest rate $(r(t))_{t \in [0, T]}$:

$$
r(t) = r_0(t) + \int_0^t \int_{\mathbb{R}^d} \sigma(\xi_1, u, t-u)c(\xi_1 - \xi_2)I_\sigma(\xi_2, u, t-u)d\xi_1d\xi_2du
+ \int_0^t \int_{\mathbb{R}^d} \sigma(\xi_2, u, t-u)M^G(d\xi, du) + \int_0^t \int_{\mathbb{R}^d} \gamma(\xi, u-, t-u-)M^P(d\xi, du)
$$

Preprint ver. file: levy_field-fr-r1-20101018.tex date: October 18, 2010
which yields that Recall (4.6). The discounted process (4.10) will be used to price the interest rate derivatives. Now, we turn to consider a financing strategy, say \( \pi = (\pi(t))_{t \in [0,T]} \), which is defined as the portfolio \( \pi(t) = (\beta_0(t), \beta_1(t)) \). Here \( \beta_0(t) \) is the time-\( t \) amount of the money market account borrowed by the investor, and \( \beta_1(t) \) represents the time-\( t \) amount of the zero-coupon bond with the price \( P(t,T) \) invested by the investor. Hence, the time-\( t \) value of this portfolio is

\[
X(t) = \frac{\beta_0(t)}{D(t)} + \beta_1(t)P(t,T).
\]

If this strategy is self-financing, we have

\[
dX(t) = \beta_1(t^-)d_1P(t,T) + (X(t) - \beta_1(t)P(t,T))r(t)dt = r(t)X(t)dt + \frac{\beta_1(t^-)}{D(t)}d(D(t)P(t,T)),
\]

which yields that

\[
d(D(t)X(t)) = \beta_1(t^-)d(D(t)P(t,T)).
\]

Recall (4.6). The discounted process \( (D(t)P(t,T))_{t \in [0,T]} \) is a \( \mathbb{Q} \)-martingale and so is \( (D(t)X(t))_{t \in [0,T]} \). On the other hand, we notice that no matter how an investor invests, the return rate of this portfolio under \( \mathbb{Q} \) must be equal to the interest rate \( r(t) \). Let \( X(0) = 0 \). Then

\[
\mathbb{E}^\mathbb{Q}[D(T)X(T)] = D(0)X(0) = 0,
\]

where \( \mathbb{E}^\mathbb{Q} \) denotes the mathematical expectation operator with respect to the risk-neutral probability measure \( \mathbb{Q} \). Suppose that \( X(T) \) satisfies that \( P(X(T) \geq 0) = 1 \). Note that \( \mathbb{P} \sim \mathbb{Q} \), then \( \mathbb{Q}(X(T) < 0) = 0 \). This, combining with (4.11), implies \( \mathbb{Q}(X(T) > 0) = 0 \). Hence \( \mathbb{P}(X(T) > 0) = 0 \). This shows that the portfolio value process \( (X(t))_{t \in [0,T]} \) is non-arbitrage (see also Shreve [39]).

5. The derivative pricing

In this section, we assume that \((\mu(t, T - t))_{t \in [0,T]}, (\sigma(\xi, t, T - t))_{(\xi, t) \in \mathbb{R}^d \times [0,T]}\) and \((\gamma(\xi, t, T - t))_{(\xi, t) \in \mathbb{R}^d \times [0,T]}\) are deterministic functions. Recall that the drift condition (4.5) is assumed to be satisfied. We focus on a general payoff \( V(T, P(T, \cdot), r(T)) \), which admits the form

\[
V(T, P(T, \cdot), r(T)) = \int_T^{T_f} v(T, P(T, U), r(T))\zeta(dU),
\]

where \( \zeta(\cdot) \) is a deterministic Borel measure and the function \( v \) corresponds to some specific derivative (see, e.g., Aihara and Bagchi [1]).

Let \( t \leq T \) be fixed and \( \delta \geq 0 \). The price with the payoff \( V \), discounted by the factor \( \exp(\int_t^{T+\delta} r(s)ds) \), is given by

\[
C(t, P, r) = \mathbb{E}^\mathbb{Q} \left[ \exp \left( - \int_t^{T+\delta} r(s)ds \right) \int_T^{T_f} v(T, P(T, U), r(T))\zeta(dU) \bigg| \mathcal{F}_t \right]
\]

Preprint ver. file: levy_field-fr-r1-201010118.tex date: October 18, 2010
\[
\tilde{c}(t, U, P, r) = \mathbb{E}^q \left[ \exp \left( - \int_t^{T+\delta} r(s) ds \right) v(T, P(T, U), r(T)) \bigg| \mathcal{F}_t \right].
\]

(5.3)

Next we derive an explicit expression for \( \tilde{c}(t, U, P, r) \). To this end, define

\[
k(t, T-t) := \int_{\mathbb{R}^d} \sigma(\xi_1, t, T-t)c(\xi_1 - \xi_2)I_{\sigma}(\xi_2, t, T-t)d\xi_1 d\xi_2
\]

\[
+ \int_{\mathbb{R}^d} \gamma(\xi, T-t)(1 - e^{-I_{\gamma}(\xi, t, T-t)})\varphi(\xi, t)\nu(d\xi).
\]

(5.4)

Recall \( M^G \) and \( M^P \) respectively in (4.8) and (4.9), which are \( \mathbb{Q} \)-martingale measures. Their \( \mathbb{Q} \)-covariance functionals are respectively determined by

\[
\langle M^G(\phi_1), M^G(\phi_2) \rangle_t^q = \int_0^t \int_{\mathbb{R}^d} \phi_1(\xi, u)c(\xi_1 - \xi_2)\phi_2(\xi, u)d\xi_1 d\xi_2 du,
\]

(5.5)

and

\[
[M^P(\phi_1), M^P(\phi_2)]_t^q = \int_0^t \int_{\mathbb{R}^d} \phi_1(\xi, u)\phi_2(\xi, u)J^P(d\xi, du),
\]

(5.6)

for \( \phi_1, \phi_2 \in \mathcal{D}(\mathbb{R}^{d+1}) \). Recall (4.5). Then under the risk-neutral measure \( \mathbb{Q} \),

\[
r(t) = f_0(t) + \int_0^t k(u, t-u)du + \int_0^t \int_{\mathbb{R}^d} \sigma(\xi_2, u, t-u)M^G(d\xi_2, du)
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} \gamma(\xi, t-u)(1 - e^{-I_{\gamma}(\xi, u, T-u)})\varphi(\xi, t)\nu(d\xi).
\]

(5.7)

This yields that, for \( t \leq T \leq U \),

\[
\frac{dP^P(t, U)}{P(t, U)} = r(t)dt - \int_{\mathbb{R}^d} I_{\sigma}(\xi, t, T-t)M^G(d\xi, dt)
\]

\[
+ \int_{\mathbb{R}^d} (e^{-I_{\gamma}(\xi, t, T-t)} - 1)M^P(d\xi, dt).
\]

(5.8)
\begin{align}
+ \int_t^T \int_{\mathbb{R}^d} (1 - e^{-L_s(\xi, u^-, T-u^-)}) \varphi(\xi, u) \nu(d\xi, du). \quad (5.9)
\end{align}

The following proposition presents an integral representation for the instantaneous interest rate \( r(t) \) \( t \in [0, T] \).

**Proposition 5.1.** Under the risk-neutral measure \( Q \), it holds that
\[
\int_t^T r(s) ds = \int_t^T \left( f(t, s) + \int_t^s k(u, s-u) du \right) ds + \int_t^T \int_{\mathbb{R}^d} I_\sigma(\xi, u, T-u) M^G(d\xi, du) \\
+ \int_t^T \int_{\mathbb{R}^d} I_\gamma(\xi, u^-, T-u^-) M^P(d\xi, du). \quad (5.10)
\]

and at the maturity \( T \leq T_f \),
\[
r(T) = f(T, T) + \int_T^T k(u, T-u) du + \int_T^T \int_{\mathbb{R}^d} \sigma(\xi, u, T-u) M^G(d\xi, du) \\
+ \int_T^T \int_{\mathbb{R}^d} \gamma(\xi, u^-, T-u^-) M^P(d\xi, du). \quad (5.11)
\]

**Proof.** From the interest rate dynamics (5.7), it follows that
\[
\int_t^T r(s) ds = \int_t^T \left( f_0(s) + \int_t^s k(u, s-u) du \right) ds + \int_t^T \left( \int_t^s \int_{\mathbb{R}^d} \sigma(\xi, u, s-u) M^G(d\xi, du) \right) ds \\
+ \int_t^T \left( \int_t^s \int_{\mathbb{R}^d} \gamma(\xi, u^-, s-u^-) M^P(d\xi, du) \right) ds \\
= \int_t^T \left( f(t, s) + \int_t^s k(u, s-u) du \right) ds + \int_t^T \left( \int_t^s \int_{\mathbb{R}^d} \sigma(\xi, u, s-u) M^G(d\xi, du) \right) ds \\
+ \int_t^T \left( \int_t^s \int_{\mathbb{R}^d} \gamma(\xi, u^-, s-u^-) M^P(d\xi, du) \right) ds. \quad (5.12)
\]

Thus the Stochastic Fubini Theorem shows the validity of (5.10). Note that, under the risk-neutral measure \( Q \),
\[
r(T) = f(T, T) \\
= f_0(T) + \int_0^T k(u, T-u) du + \int_0^T \int_{\mathbb{R}^d} \sigma(\xi, u, T-u) M^G(d\xi, du) \\
+ \int_0^T \int_{\mathbb{R}^d} \gamma(\xi, u^-, T-u^-) M^P(d\xi, du). \quad (5.13)
\]

Then, the integral representation (5.11) follows from (3.3). Thus the proof of the proposition is completed.

We next turn to (5.3). By virtue of Proposition 5.1, we have, for \( U \geq T \),
\[
\bar{c}(t, U, P, r) = \exp \left( - \int_t^{T+\delta} \left( f(t, s) + \int_t^s k(u, s-u) du \right) ds \right)
\]
\[ e(t, T) := v\left( T, \exp\left( \int_0^T \left( f(t, s) + \int_s^T k(u, s-u) du \right) ds + \int_t^T \int_{\mathbb{R}^d} I_\sigma(\xi, u, T - u) M^G(d\xi, du) \right) \right) 
+ \int_t^T \int_{\mathbb{R}^d} I_\gamma(\xi, u^-, T - u^-) M^P(d\xi, du) - \int_t^T \int_{\mathbb{R}^d} I_\sigma(\xi, u, U - u) M^G(d\xi, du) 
- \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} I_\sigma(\xi_1, u, U - u) c(\xi_1 - \xi_2) I_\sigma(\xi_2, u, U - u) d\xi_1 d\xi_2 du 
- \int_t^T \int_{\mathbb{R}^d} I_\gamma(\xi, u^-, U - u^-) J^P(d\xi, du) 
+ \int_t^T \int_{\mathbb{R}^d} \left( 1 - e^{-I_\gamma(\xi, u^-, U - u^-)} \right) \varphi(\xi, u) \nu(d\xi) du \right) P(t, U), \]
\[ f(t, T) + \int_t^T k(u, T - u) du + \int_t^T \int_{\mathbb{R}^d} \sigma(\xi, u, T - u) M^G(d\xi, du) 
+ \int_t^T \int_{\mathbb{R}^d} \gamma(\xi, u^-, T - u^-) M^P(d\xi, du) \right). \]

Note that the random field \( M^P(d\xi, du) \) is a \( \mathbb{Q} \)-compensated Poisson random measure. Then, for any deterministic integrand \( g \in \Psi \),

\[
\mathbb{E}^q\left[ \exp\left( \int_0^t \int_{\mathbb{R}^d} g(\xi, u) M^P(d\xi, du) \right) \right] 
= \exp\left( \int_0^t \int_{\mathbb{R}^d} (e^{g(\xi, u)} - 1 - g(\xi, u)) \varphi(\xi, u) \nu(d\xi) du \right), \quad t \leq T_f.
\]

Hence

\[
\mathbb{E}^q\left[ \exp\left( - \int_T^{T+\delta} \int_{\mathbb{R}^d} I_\sigma(\xi, u, T + \delta - u) M^G(d\xi, du) 
- \int_T^{T+\delta} \int_{\mathbb{R}^d} I_\gamma(\xi, u^-, T + \delta - u^-) M^P(d\xi, du) \right) \right] 
= \exp\left( \frac{1}{2} \left( \langle M^G(I_\sigma^{T+\delta}) \rangle_T^{q} - \langle M^G(I_\sigma^{T+\delta}) \rangle_T \right) \right) 
+ \int_T^{T+\delta} \int_{\mathbb{R}^d} I_\gamma(\xi, u, T + \delta - u) \varphi(\xi, u) \nu(d\xi) du
\]
\[
+ \int_T^{T+\delta} \int_{\mathbb{R}^d} \left( e^{-I_i(\xi,u,T+\delta-u)} - 1 \right) \varphi(\xi,u)\nu(d\xi)\nu(du),
\]

where \( I_i^T(\xi,u) = I_i(\xi,u,T-u) \), for \( i \in \{\sigma, \gamma\} \). For \( \hat{t} \geq t \), we introduce the following notations

\[
\hat{c}(t,T,\delta; f, k, \sigma, \gamma, \varphi, \nu)
:= \exp \left( -\int_T^{T+\delta} (f(t,s) + \int_s^t k(u,s-u)du)ds \right)
\times \exp \left( \frac{1}{2} \left( \left( \mathbb{E}^G(I_{\sigma}^{T+\delta}) \right)_T^q - \left( \mathbb{E}^G(I_{\sigma}^{T+\delta}) \right)_T^q \right) \right)
+ \int_T^{T+\delta} \int_{\mathbb{R}^d} I_\sigma(\xi,u,T+\Delta-u)\varphi(\xi,u)\nu(d\xi)du
+ \int_T^{T+\delta} \int_{\mathbb{R}^d} (e^{-I_\sigma(\xi,u,T+\delta-u)} - 1)\varphi(\xi,u)\nu(d\xi)du,
\]

\[
X(\hat{t},t) := \exp \left( -\int_{\hat{t}}^{\hat{t}} \int_{\mathbb{R}^d} I_\sigma(\xi,u,T+\delta-u)M^G(d\xi,du) \right.
\left. - \int_{\hat{t}}^{\hat{t}} \int_{\mathbb{R}^d} I_\gamma(\xi,u^-T+\delta-u^-)M^P(d\xi,du) \right),
\]

\[
\log \frac{Y(\hat{t},t)}{Y(t,t)} := \int_t^{\hat{t}} \int_{\mathbb{R}^d} I_\sigma(\xi,u,T-u)M^G(d\xi,du) + \int_t^{\hat{t}} \int_{\mathbb{R}^d} I_\gamma(\xi,u,T-u)M^P(d\xi,du)
- \int_t^{\hat{t}} \int_{\mathbb{R}^d} I_\sigma(\xi,u,U-u)M^G(d\xi,du) - \frac{1}{2} \left( \left( \mathbb{E}^G(I_{\sigma}^{U}) \right)_T^q - \left( \mathbb{E}^G(I_{\sigma}^{U}) \right)_T^q \right)
- \int_t^{\hat{t}} \int_{\mathbb{R}^d} I_\gamma(\xi,u^-,U-u^-)M^P(d\xi,du)
+ \int_t^{\hat{t}} \int_{\mathbb{R}^d} (1-e^{-I_\sigma(\xi,u^-,U-u^-)})\varphi(\xi,u)\nu(d\xi)du,
\]

and

\[
R(\hat{t},t) := f(t,T) + \int_t^{T} k(u,T-u)du + \int_t^{\hat{t}} \int_{\mathbb{R}^d} \sigma(\xi,u,T-u)M^G(d\xi,du)
+ \int_t^{\hat{t}} \int_{\mathbb{R}^d} \gamma(\xi,u^-,T-u^-)M^P(d\xi,du),
\]

where, for \( t < T \),

\[
Y(t,t) = \exp \left( \int_t^{T} (f(t,s) + \int_s^t k(u,s-u)du)ds \right) P(t,U).
\]
Combining (5.14) with (5.16), we have
\[ \bar{c}(t, U, P, r) = \bar{c}(t, T, \Delta; f, k, \sigma, \gamma, \varphi, \nu) \mathbb{E}^I \left[ X(T, t) u(T, Y(T, t), R(T, t)) | \mathcal{F}_t \right]. \]

Moreover, we have the following

**Proposition 5.2.** The random fields \( X(\hat{t}, t), Y(\hat{t}, t) \) and \( R(\hat{t}, t) \) respectively satisfy the following stochastic differential equations (SDEs) with jumps

\[
\frac{d_t X(\hat{t}, t)}{X(t^-, t)} = -\int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

\[
\frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

\[
\frac{d_t Y(\hat{t}, t)}{Y(t^-, t)} = \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

\[
\frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

and

\[
\frac{d_t R(\hat{t}, t)}{R(t^-, t)} = \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

\[
\frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

Moreover, we have

\[
\frac{d_t X(\hat{t}, t) d_t Y(\hat{t}, t)}{X(t^-, t) Y(t^-, t)} = \frac{d}{d_t} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

\[
\frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

and

\[
\frac{d_t Y(\hat{t}, t) d_t R(\hat{t}, t)}{Y(t^-, t) R(t^-, t)} = \frac{d}{d_t} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]

\[
\frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \left[ \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{G}}(\xi, \mathbb{F}) \right] d\xi, dt
\]
We further introduce the following notations.

\[ \begin{align*}
\hat{a}(t, T, U) &= \frac{d \langle M^G(I_q^T), M^G(I_q^U) \rangle_t}{dt} \\
\hat{b}(t, T, U) &= \int_{\mathbb{R}^d} (1 - e^{-I_q(\xi,t^-T^-t^-)}) (1 - e^{-I_q(\xi,t^-,U-t^-)}) \varphi(\xi, t) \nu(d\xi), \\
\hat{c}(t, T, U) &= \int_{\mathbb{R}^d} (1 - e^{-I_q(\xi,t^-T^-t^-)}) I_\gamma(\xi, t^-U - t^-) \varphi(\xi, t) \nu(d\xi), \\
\hat{d}(t, T, U) &= \int_{\mathbb{R}^d} I_\gamma(\xi, t^-T^-t^-) I_\gamma(\xi, t^-U - t^-) \varphi(\xi, t) \nu(d\xi), \\
\hat{e}(t, T) &= \int_{\mathbb{R}^d} (1 - e^{-I_q(\xi,t^-T^-t^-)}) \varphi(\xi, t) \nu(d\xi), \\
\hat{f}(t, T) &= \int_{\mathbb{R}^d} I_\gamma(\xi, t^-T^-t^-) \varphi(\xi, t) \nu(d\xi),
\end{align*} \]

and

\[ \begin{align*}
a_{11}(\hat{t}, T + \delta) &= \hat{a}(\hat{t}, T + \delta, T + \delta) + \hat{b}(\hat{t}, T + \delta, T + \delta), \\
a_{22}(\hat{t}, T, U) &= \hat{a}(\hat{t}, T, T) + \hat{a}(\hat{t}, U, U) - \hat{a}(\hat{t}, T, U) + \hat{b}(\hat{t}, T, T) + \hat{b}(\hat{t}, U, U) + \hat{b}(\hat{t}, T, U), \\
a_{33}(\hat{t}, T) &= \hat{a}(\hat{t}, T, T) + \hat{d}(\hat{t}, T, T), \\
a_{12}(\hat{t}, T, U + \delta) &= \hat{a}(\hat{t}, U, T + \delta) - \hat{a}(\hat{t}, T + \delta) + \hat{b}(\hat{t}, U, T + \delta) + \hat{b}(\hat{t}, T, T + \delta), \\
a_{13}(\hat{t}, T, T + \delta) &= -\hat{a}(\hat{t}, T, T + \delta) + \hat{c}(\hat{t}, T + \delta, T), \\
a_{23}(\hat{t}, T, U) &= \hat{a}(\hat{t}, T, T) - \hat{a}(\hat{t}, U, T) - \hat{c}(\hat{t}, T, T) - \hat{c}(\hat{t}, U, T), \\
a_1(\hat{t}, T + \delta) &= \frac{1}{2} \hat{a}(\hat{t}, T + \delta, T + \delta) + \hat{c}(\hat{t}, T + \delta, T) - \hat{c}(\hat{t}, T, T) - \hat{f}(\hat{t}, T + \delta), \\
a_2(\hat{t}, T) &= \frac{1}{2} \hat{a}(\hat{t}, T, T) - \hat{c}(\hat{t}, T) + \hat{f}(\hat{t}, T).
\end{align*} \]

Finally, we state the main result of this section.

**Theorem 5.1.** The price kernel (5.3) of the contingent claim functional \( V \) given by (5.1) can be evaluated as follows

\[ \begin{align*}
\tilde{c}(t, U, P, r) &= \tilde{c}(t, T, \delta; f, k, \sigma, \gamma, \varphi, \nu) \\
&\times V \left( t, 1, \exp \left( \int_t^T (f(t, s) + \int_s^T k(u, s - u) du) ds \right) P(t, U), \right.
\left. f(t, T) + \int_t^T k(u, T - u) du \right).
\end{align*} \]
where the payoff function \( V(t, x, y, r) \in C^{1,2}([0, T] \times \mathbb{R}^3) \) satisfies the following parabolic PDE

\[
\frac{\partial V(\hat{t}, x, y, r)}{\partial \hat{t}} + A_{\hat{t},x,y,r}V(\hat{t}, x, y, r) = 0, \quad (\hat{t}, x, y, r) \in (0, T) \times \mathbb{R}^3,
\]

\[(5.17)\]

with the differential operator

\[
A_{\hat{t},x,y,r} = \frac{1}{2} x^2 a_{11}(\hat{t}, T, U) \frac{\partial^2}{\partial x^2} + \frac{1}{2} y^2 a_{22}(\hat{t}, T, U) \frac{\partial^2}{\partial y^2} + \frac{1}{2} a_{33}(\hat{t}, T) \frac{\partial^2}{\partial r^2} + xy a_{12}(\hat{t}, T, U) \frac{\partial}{\partial x} + xa_1(\hat{t}, T) \frac{\partial}{\partial x} + ya_2(\hat{t}, T) \frac{\partial}{\partial y}.
\]

Proof. The desired results follows from a direct application of the Itô formula with jumps to the payoff functional

\[
V(\hat{t}, X(\hat{t}, t), Y(\hat{t}, t), R(\hat{t}, t)).
\]

A special case is that, if \( \bar{c}(t, U, P, r) = \bar{c}(t, U, P) \) and set \( \check{v}(\hat{t}, x, y) = x \hat{v}(\hat{t}, y) \), then we have the PDE for \( \hat{v} \) as follows

\[
\frac{\partial \hat{v}}{\partial \hat{t}} + \frac{1}{2} \frac{a_{22} y^2}{\partial y^2} + (a_{12} + a_2)y \frac{\partial \hat{v}}{\partial y} + a_1 \hat{v} = 0, \quad (\hat{t}, y) \in (0, T) \times \mathbb{R},
\]

\[
\hat{v}(T, y) = v(T, y). \tag{5.18}
\]

Finally we present a concrete example of the model (3.2). For simplicity, let the dimension \( d = 1 \). In addition, assume a time-invariant market price of diffusive risk given by \( \psi(\xi, t) = \gamma \sqrt{2 \pi} \exp(-\frac{\xi^2}{2}) \) and a constant market price of jump risk \( \varphi(\xi, t) = \gamma \). Further, suppose the volatility field is given by

\[
\sigma(\xi, t, T-t) = \frac{\sigma(T-t)}{\sqrt{2 \pi}} \exp\left(-\frac{\xi^2}{2}\right),
\]

the jump volatility field \( \gamma(\xi, t, T-t) = \gamma \), and the correlated kernel is a constant \( \bar{c} \), which obviously belongs to \( \Lambda_c \). Let \( J^P(\cdot, \cdot, \cdot) \) be a Poisson process with intensity \( \lambda > 0 \). Then, the associated jump measure \( \nu(d\xi) = \lambda \delta_1(d\xi) \in \Lambda_\nu \). Recall that

\[
I_i(\xi, t, T-t) = \int_t^T i(\xi, t, y-t)dy, \quad \text{for} \quad i \in \{\sigma, \gamma\}.
\]

Then, the no-arbitrage drift condition (4.5) is reduced to

\[
\mu(t, T-t) = \bar{c}\sigma^2 \frac{(T-t)^3}{2} + \bar{c}\gamma \psi(T-t) + \gamma(1 - \gamma e^{-\gamma(T-t)}). \tag{5.19}
\]
From (5.4), it concludes that
\[
k(t, T - t) = \bar{c}\sigma^2 (T - t)^3 + \gamma \gamma_p (1 - e^{-\gamma(T-t)}).
\] (5.20)
Thus, all coefficients of the operator \( A_{t,x,y,r} \) in (5.17) can be expressed explicitly. If we consider the caplet with reset date \( T \) and settlement date \( T + \delta \) that pays the holder the difference between the LIBOR rate and the strike \( K \), the boundary condition in (5.18) will be
\[
\hat{v}(T, y) = \left( \frac{1}{y} - 1 - \delta K \right)^+, \quad y > 0.
\] (5.21)
Now the PDEs (5.17) and (5.18) with the above boundary condition can be solved numerically (see, e.g., Duffie [11]).

6. Conclusions
In this paper, under the HJM framework, we developed a forward rate model driven by a Lévy random field. The Lévy random field is composed of a kernel-correlated Gaussian random field and a Poisson random field. We introduced the stochastic integral with respect to the Lévy field and then described the forward rate model. By using the Itô formula with jumps, a non-arbitrage condition that can be completely characterized by the drift of the forward rates was obtained. After that, we identified a risk-neutral probability measure. Finally, the pricing for some interest rate derivatives with a general payoff functional was derived via a related PDE argument, and a concrete example that can be solved numerically was also presented.

Acknowledgements
This work was supported by the LPMC at Nankai University and the Keygrant Project of Chinese Ministry of Education (No. 309009). The research of Bo was also supported by the NSF of China (No. 11001213).

References


Preprint ver. file: levy_field-fr-r1-20101018.tex date: October 18, 2010