

Reflected Diffusion Model for Exchange Rates in a Target Zone

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Introduction

Target zones for exchange rates have been a reality in the European Monetary System (EMS) and many other monetary systems (see e.g. Lundbergh and Teräsvirta (2006)), in which an exchange rate dynamics was promised to remain within a fixed zone.

Since the seminal work by Krugman (1991), various tractable models have been proposed to describe the exchange rates restricted in a target zone. De Jong, Drost and Werker (2001) introduced a mean-reverting jump-diffusion model for the exchange rate in a target zone which was inspired by a modified two-limit version of CIR model. In a recent paper, Larsen and Sørensen (2007) generalized the model to allow the asymmetry between currencies.

Introduction

In the last two models, the exchange rate was in some band due to the variance decreasing as the exchange rate approaches the upper or the lower boundaries of the band. However, Larsen and Sørensen (2007) also noted that an alternative model with a constant volatility may be more realistic, since the random variation in the exchange rate data is not smaller near the boundaries of the target zone than near the central parity. This insight shall be incorporated into our model.

The Reflected O-U Model

Let $W = \{W_t, t \geq 0\}$ be a standard Wiener process. We model the deviation of the exchange rate from the central parity in percent by the following reflected Ornstein-Uhlenbeck (ROU) process with two sided barriers,

$$X_t = x_0 + \alpha \left(\gamma t - \int_0^t X_s ds \right) + \sigma W_t + L_t - U_t,$$
$$X_t \in [b^L, b^U], \text{ for all } t \geq 0,$$

where x_0 denotes the initial state value, $\alpha > 0$ is the speed of reversion, $\sigma > 0$ is the volatility, and $\gamma \in (b^L, b^U)$ is the asymmetric parameter which admits a similar meaning as in Larsen and Sørensen (2007).

The Reflected O-U Model

$L.$ and $U.$ are nondecreasing and continuous with $L_0 = U_0 = 0$,

$$\int_0^t \mathbf{1}_{\{X_s > b^L\}} dL_s = \int_0^t \mathbf{1}_{\{X_s < b^U\}} dU_s = 0, \quad \text{for all } t \geq 0.$$

Roughly speaking, L and U increase in the minimal amounts sufficient to ensure $X_t \in [b^L, b^U]$ for all $t \geq 0$ (see Harrison (1986) and Ata, Harrison and Shepp (2005)). It is known that the solution triple (X, L, U) admits the continuous path modification and strong Markov property.

Parameter Estimation

Suppose that we observe the process at discrete dates

$$\{t_i = (i - 1)\Delta : i = 1, \dots, k\},$$

where $\Delta > 0$ is fixed. Let $p(\Delta, x, y; \theta)$ be the conditional density of $X_{t+\Delta} = y$ given $X_t = x$ induced by X . Then the log-likelihood function has the simple form (see e.g. Aït-Sahalia (2002)),

$$\ell_k(\theta) := \sum_{i=1}^{k-1} \ln\{p(\Delta, X_{(i-1)\Delta}, X_{i\Delta}; \theta)\},$$

where $\theta = (\alpha, \gamma, \sigma)^T$ denotes the parameter vector.

Parameter Estimation

The transition density for ROU processes is unknown explicitly. While it is known that conditioned on $L_t - L_s = U_t - U_s = 0$ with $0 \leq s < t < \infty$, X_t is just the O-U process on the time interval (s, t) and most of the path is in the interior of the zone. Intuitively, we may replace the transition density of ROU process by that of the corresponding O-U process, i.e.,

$$p(t, x, y; \theta) = \frac{\sqrt{\alpha}}{\sqrt{\pi(1 - e^{-2\alpha t})}\sigma} \exp\left(-\frac{\alpha[y - xe^{-\alpha t} - \gamma(1 - e^{-\alpha t})]^2}{(1 - e^{-2\alpha t})\sigma^2}\right),$$

in the likelihood function.

Parameter Estimation

However, the MLE of these three parameters are not satisfactory, we have to derive an alternative scheme which fits our situation. The scheme is to establish estimators $\hat{\alpha}$ and $\hat{\gamma}$ of α and γ , respectively, which are independent of σ , then use MLE to estimate σ , i.e.,

$$\hat{\sigma} = \arg \max_{\sigma > 0} \ell_k(\sigma; \hat{\alpha}, \hat{\gamma}).$$

Next, We will find $\hat{\alpha}$ and $\hat{\gamma}$.

Parameter Estimation: Estimator of γ

Consider the following reflected diffusion with barriers b^L and b^U ,

$$dX_t = \mu(X_t)dt + \sigma dW_t + dL_t - dU_t,$$

and here we assume the parameter $\sigma > 0$ is known. Under the stationary assumption, it holds that

$$\mu(x) = \frac{\sigma^2 \pi'(x)}{2\pi(x)},$$

where $\pi(x)$ denotes the stationary density of X .

Parameter Estimation: Estimator of γ

We estimate $\pi(x)$ by the kernel estimator $\hat{\pi}$ given by

$$\hat{\pi}(x) = \frac{1}{kh} \sum_{i=0}^k K\left(\frac{x - X_{t_i}}{h}\right),$$

where K is a kernel and h is the bandwidth. The non-parametric estimator $\hat{\mu}(x)$ of $\mu(x)$ goes as follows,

$$\hat{\mu}(x) = \frac{\sigma^2 \hat{\pi}'(x)}{2\hat{\pi}(x)}.$$

Parameter Estimation: Estimator of γ

To estimate the parameters α and γ , we use the following optimization criterion,

$$(\bar{\alpha}, \bar{\gamma}) = \arg \min_{(\alpha, \gamma) \in [0, \infty \times [b^L, b^U]]} \|\hat{\mu}(x) - \alpha(\gamma - x)\|,$$

where $\|\cdot\|$ denotes the corresponding norm of Banach space $C([b^L, b^U])$.

As an example, we can choose the norm $\|\cdot\|$ in (0.1) as the usual norm on $L^p([b^L, b^U], \nu)$ with $1 \leq p \leq \infty$, i.e.,

$$\|f\| = \begin{cases} \left(\int_{[b^L, b^U]} |f(x)|^p \nu(dx) \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_{x \in [b^L, b^U]} |f(x)|, & p = \infty. \end{cases}$$

Here ν is a finite measure on $[b^L, b^U]$.

Parameter Estimation: Estimator of γ

When $p = 2$ and $\nu(dx) = dx$, i.e., the Lebesgue measure, we can express the estimator of α and γ explicitly as

$$\hat{\alpha} = \frac{\sigma^2}{2} \left[\frac{6(b^L + b^U)}{(b^U - b^L)^3} \int_{b^L}^{b^U} \frac{\hat{\pi}'(x)}{\hat{\pi}(x)} dx - \frac{12}{(b^U - b^L)^3} \int_{b^L}^{b^U} x \frac{\hat{\pi}'(x)}{\hat{\pi}(x)} dx \right],$$
$$\hat{\gamma} = \frac{4 [(b^L)^2 + b^L b^U + (b^U)^2] \int_{b^L}^{b^U} \frac{\hat{\pi}'(x)}{\hat{\pi}(x)} dx - 6(b^L + b^U) \int_{b^L}^{b^U} x \frac{\hat{\pi}'(x)}{\hat{\pi}(x)} dx}{6(b^L + b^U) \int_{b^L}^{b^U} \frac{\hat{\pi}'(x)}{\hat{\pi}(x)} dx - 12 \int_{b^L}^{b^U} x \frac{\hat{\pi}'(x)}{\hat{\pi}(x)} dx}.$$

Thereafter, we shall use the estimator $\hat{\gamma}$ to estimate γ . Since the estimator $\hat{\alpha}$ still depends on the value of σ , we have to seek another one.

Parameter Estimation: Estimator of α

We use the estimator of α based on the continuous observations $\{X_t; 0 \leq t \leq T\}$:

$$\hat{\alpha}_T = \frac{\int_0^T (\gamma - X_t) dX_t + \int_0^T (\gamma - X_t) dU_t - \int_0^T (\gamma - X_t) dL_t}{\int_0^T (\gamma - X_t)^2 dt}.$$

For this estimator, we have

Theorem

The estimator $\hat{\alpha}_T$ of α admits strongly consistency and asymptotic normality as $T \rightarrow \infty$.

Parameter Estimation: Estimator of α

Note that the estimator $\hat{\alpha}$ is based on continuous observations. It is not consistent with our setting. So we have to put forward a reasonable estimator which is in line with our context. It is known that the large majority of the paths of ROU processes is situated inside the interval (b^L, b^U) , which means that we can ignore the regulators L and U in the computation (another reason is that it is practically impossible to find these regulators from the discrete observations of X). Thus we set

$$\hat{\alpha} = \frac{\sum_{i=1}^{k-1} (\gamma - X_{t_i})(X_{t_{i+1}} - X_{t_i})}{\frac{1}{\Delta} \sum_{i=1}^{k-1} (\gamma - X_{t_i})^2}.$$

Simulation Study

We study the finite sample properties of the estimate scheme. Here we take the true parameter values as $\alpha = 0.0641523$, $\gamma = 0.0837838$, $\sigma = 0.0764972$ and the boundaries as $b^L = -0.4285715$, $b^U = 0.5982143$, which are the same as the estimations we obtain from the the exchange rate index of the Norwegian krone (which contains 449 observations). The time between the sampling points is 1. We perform 100 Monte Carlo simulations of the sample paths generated by the model, each containing 200, 449, 800 observations respectively. The estimation of σ looks fine, but the estimator of γ possesses a relatively large standard deviation. The estimator of α is comparable to the existing literature.

Simulation Study

Mean and Standard Error of the estimators.

*The true parameter values are $\alpha = 0.0641523$, $\gamma = 0.0837838$, $\sigma = 0.0764972$,
 $b^L = -0.4285715$ and $b^U = 0.5982143$.*

NoO	200	449	800
Mean $\hat{\alpha}$	0.0760897	0.0710042	0.0672419
SE $\hat{\alpha}$	0.0247234	0.0150568	0.0122737
Mean $\hat{\gamma}$	0.0773711	0.0895498	0.0791298
SE $\hat{\gamma}$	0.0968416	0.0729897	0.0508196
Mean $\hat{\sigma}$	0.0756266	0.0762249	0.0759344
SE $\hat{\sigma}$	0.0040170	0.0027279	0.0021366

Simulation Study

To show the validity of the estimator $\hat{\gamma}$ for large γ , we perform another Monte Carlo simulation.

Mean and Standard Error of the estimator $\hat{\gamma}$.

The true parameter values are $\alpha = 0.1$, $\gamma = 0.5$, $\sigma = 0.1$,

$$b^L = 0 \text{ and } b^U = 1.$$

NoO	200	400	800
Mean $\hat{\gamma}$	0.4930712	0.4957451	0.4974663
SE $\hat{\gamma}$	0.0776183	0.0553992	0.0371763

Remark

The parameter estimate scheme is suitable (notamiss) for our setting.

The Norwegian krone: 1989-1990

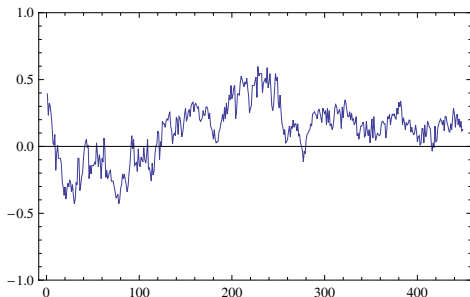
The Norwegian exchange rate index studied in the paper covers the period from January 2, 1989 to October 22, 1990, in total 449 observations. In order to estimate the stationary density, we have to insure that the exchange rate systems are in the steady state in this paper, so we perform the augmented Dickey-Fuller non-stationary test. Note that the Norwegian krone (1989-1990) reject the the unit root hypothesis at 95% level.

The augmented Dickey-Fuller non-stationary test.

Currencies	observations range (years)	<i>p</i> -values
The Norwegian krone	1989–1990	0.0435

The Norwegian krone: 1989-1990

A graph of the index's deviation from the central parity in percent is presented in the following Figure.



The Norwegian krone: 1989-1990

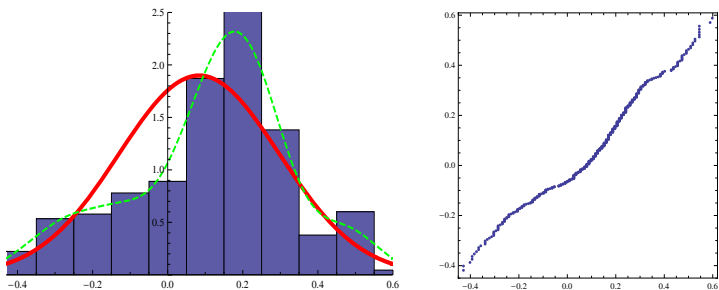
It is known that Norges Bank started to maintain an implicit target zone that was narrower than the official one in the late of 1988. Herein, we take

$$b^L = \min\{X_{t_i} :, i = 1, \dots, k\}, \quad b^U = \max\{X_{t_i} :, i = 1, \dots, k\}.$$

Mean and Standard Error of the estimated parameters for the Norwegian krone exchange rate index.

Est.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\sigma}$	b^L	b^U
Mean	0.0641523	0.0837838	0.0764972	-0.4285715	0.5982143
SE	0.0150568	0.0729897	0.0027279	—	—

The fit of the stationary density



The left panel is the histogram of the Norwegian exchange rate index. The dashed line is the Gaussian kernel estimation of the density for the index; The solid line is the stationary density of the ROU process with the estimated parameters. The right panel is the PP plots comparing the estimated stationary distribution to the Norwegian krone exchange rate index.

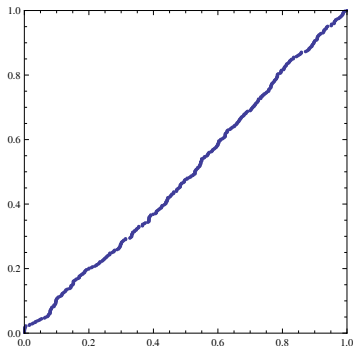
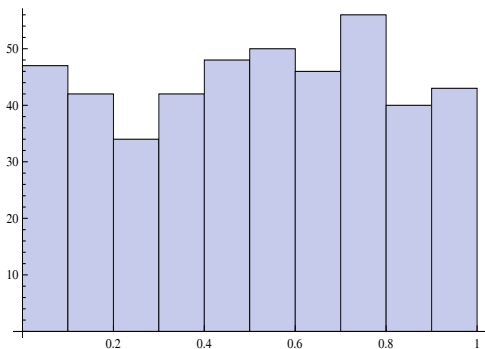
The fit of the transition distribution

To study the fit of the transition distribution, we use the simulated uniform residuals proposed by Pedersen (1994), which are also used in Larsen and Sørensen (2007). The residuals are given by

$$U_i = F(X_{t_i} | X_{t_{i-1}}; \alpha, \gamma, \sigma),$$

where $F(y|x; \alpha, \gamma, \sigma)$ denotes the conditional distribution function of X_{Δ} at y given that $X_0 = x$ and the data are $\{X_{t_i} : i = 1, 2, \dots, 449\}$. If the data have been generated by ROU process with the estimated parameter values, the U_i -s are independent and uniformly distributed in the unit interval.

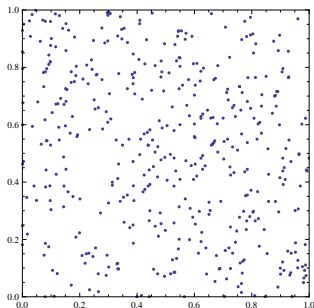
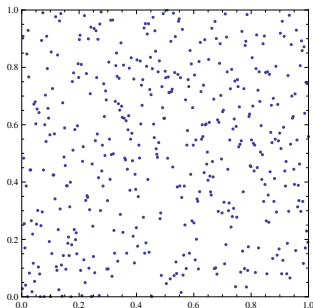
The fit of the transition distribution



The left panel is the histogram of the uniform residuals. The right panel is the PP plots comparing the empirical distribution of the uniform residuals to the uniform distribution on $[0, 1]$.

The fit of the transition distribution

The following plots show no particular pattern that contradicts the independence of the residuals.



The left panel exhibits the points $(\frac{i}{448}, U_i)$, $i = 1, 2, \dots, 448$ and the right panel exhibits the points (U_i, U_{i+1}) , $i = 1, 2, \dots, 447$.






The fit of the transition distribution

We perform the formal tests of Null hypothesis that the residuals are uniformly distributed on $[0, 1]$. Four test statistics were calculated, the χ^2 goodness-of-fit test statistics with the interval $[0, 1]$ divided into thirteen intervals, the Kolmogorov-Smirnov test statistics, the Cramer-Von Mises test statistics, and the test statistic $-2 \sum_{i=1}^n \log(U_i)$ which is χ^2 -distributed with $2n$ degrees of freedom if the residuals are uniform and independent.






The p-values of the tests for uniformity on $[0, 1]$ of residuals.

χ^2	$K - S$	$C - V$	$-2 \sum \log(U_i)$
64.3%	> 25%	> 25%	57.6%

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Thank You!