On the hitting time density for reflected OU processes: with an application to the regulated market*

Yingjie Li[1] **Yongjin Wang**[2] **Xuewei Yang**[1,3,†]

[1] *School of Mathematical Sciences, Nankai University, Tianjin 300071, China*

[2] *School of Business, Nankai University, Tianjin 300071, China*

[3] *Department of Mathematics, University of Illinois, Urbana, IL 61801, USA*

Abstract: In this paper, we derive explicit analytical densities of first hitting time for reflected Ornstein-Uhlenbeck processes by employing the eigenfunction expansion technique. An application to the regulated market is given for further illustrations.

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1. Introduction

In this paper, we derive the analytical expressions for first hitting time densities for reflected Ornstein-Uhlenbeck processes in terms of the corresponding Sturm-Liouville eigenfunction expansions. The reflected Ornstein-Uhlenbeck (ROU) processes have been studied in economics, queueing, finance, etc. Among others, Goldstein and Keirstead [4] used the ROU processes to model spot interest rate. Ward and Glynn [17–19] proved that the ROU processes can be viewed as approximations of queueing systems with reneging or balking, and some interesting properties of ROU processes were also studied by the authors. Recently, Bo et al. [1] and Bo, Wang and Yang [2] used the ROU processes to model the price dynamics in a defaultable regulated market, and the conditional default probability was obtained.

The study of the first hitting time of ROU processes is important both in itself and for its applications in queueing and finance (see, e.g., [1, 2, 17–19]). In fact, the Laplace transform of the first hitting time has been obtained in [2] and [3], and the hitting time density was obtained in [2] by using numerical Laplace inversion. In this paper, we intend to provide an alternative approach, that is, we will adopt the spectral expansion approach to diffusions

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†Corresponding author. Email: xwyangnk@yahoo.com.cn

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(see, e.g., [7, 9–12, 14, 16]) to compute the eigenfunction expansions for hitting time density for ROU processes in terms of Hermite function and confluent hypergeometric function. The large-n asymptotics of the eigenvalues and the expansion coefficients are given in terms of elementary functions. We also provide an application to the so-called regulated market, in which some numerical results are included.

The outline of the paper is as follows: Section 2 derives the explicit analytical expressions for the distribution and density for first hitting time of ROU processes. Section 3 presents an application to the regulated market and section 4 concludes.

2. Hitting time density for ROU process

In this section we consider the density of first hitting time for ROU process. Let $X = \{X_t, t \geq 0\}$ be an one dimensional ROU process with barriers 0 and 1,[1] that is:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t - \mathrm{d}U_t, \quad X_0 = x \in [0, 1], \tag{2.1}$$

where $W = \{W_t, t \geq 0\}$ is an one-dimensional standard Brownian motion, and $\theta \in (0, 1)$, $\kappa, \sigma \in (0, +\infty)$. Here $L = \{L_t, t \geq 0\}$ and $U = \{U_t, t \geq 0\}$ are the regulators at the points 0 and 1 respectively, which are uniquely determined by the following properties:

(i) Both $t \to L_t$ and $t \to U_t$ are continuous processes with $L_0 = U_0 = 0$ and $t \in \mathbb{R}^+$.

(ii) L and U are minimum nondecreasing processes such that $X \in [0, 1]$, and they satisfy $\int_0^t 1_{\{X_s=0\}} dL_t = L_t$ and $\int_0^t 1_{\{X_s=1\}} dU_t = U_t$, for $t \geq 0$.

For the existence and uniqueness of the strong solutions to (2.1), refer to [13].

The infinitesimal generator for the ROU process in (2.1) is:

$$\mathscr{A} f(x) := \kappa(\theta - x) f'(x) + \frac{\sigma^2}{2} f''(x), \quad x \in (0, 1), \tag{2.2}$$

with boundary conditions[2]:

$$f'(0) = 0, \quad \text{and} \quad f'(1) = 0. \tag{2.3}$$

Let $x \in [0, 1]$ be the starting point of the diffusion. Define the first hitting time

$$T_y := \inf\{t \geq 0 : X_t = y\}, \text{ for some fixed } y \in [0, 1]. \tag{2.4}$$

In this paper we focus on the analytic form of the density of the first hitting time,[3]

$$f_{T_y}(t; x) = \frac{P_x(T_y \in \mathrm{d}t)}{\mathrm{d}t}.$$

Then we have the following proposition.

[1] The choice of the reflecting barriers is just for simplicity, and the general cases are similar.

[2] It is easy to verify that both of the end points are regular instantaneously reflecting (see Table 15.6.2 in Karlin and Taylor [6]).

[3] In fact, the following Proposition 2.1 also proves the existence of the density. For the existence of the hitting time density for general diffusions, one may refer to [7] and the references therein.

Proposition 2.1. *Suppose X_t is the ROU process defined in (2.1), and T_y is the first hitting time of X_t defined in (2.4). For fixed $x, y \in [0,1]$ satisfying $x \neq y$, we have*

$$P_x(t < T_y) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}, \qquad t > 0, \quad \text{and} \tag{2.5}$$

$$f_{T_y}(t;x) = \sum_{n=1}^{\infty} c_n \lambda_n e^{-\lambda_n t}, \qquad t > 0, \tag{2.6}$$

where $\{\lambda_n\}_{n=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$ as $n \to \infty$, and $\{c_n\}_{n=1}^{\infty}$ are explicitly given below. Moreover, for all $t_0 > 0$, the series (2.6) converges uniformly on $[t_0, \infty)$.

(i) ***Hitting down*** $(0 \le y < x \le 1)$. *$\{\lambda_n\}_{n=1}^{\infty}, 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \infty$ are the roots of the equation*

$$w_1(y, \lambda) = 0, \quad \text{and} \tag{2.7}$$

$$c_n = \frac{w_1(x, \lambda_n)}{\lambda_n w_{1\lambda}(y, \lambda_n)}, \quad n = 1, 2, \ldots. \tag{2.8}$$

Where

$$w_1(x,\lambda) = A * \mathsf{H}\left(\frac{\lambda}{\kappa}, (x-\theta)\frac{\sqrt{\kappa}}{\sigma}\right) + {}_1\mathsf{F}_1\left(-\frac{\lambda}{2\kappa}, \frac{1}{2}, (x-\theta)^2\frac{\kappa}{\sigma^2}\right), \tag{2.9}$$

with $A = -\frac{(\theta-1)\sqrt{\kappa}\ {}_1\mathsf{F}_1\left(1-\frac{\lambda}{2\kappa}, \frac{3}{2}, (1-\theta)^2\frac{\kappa}{\sigma^2}\right)}{\sigma \mathsf{H}\left(-1+\frac{\lambda}{\kappa}, (1-\theta)\frac{\sqrt{\kappa}}{\sigma}\right)}$. Here $w_{1\lambda}(y,\lambda)$ denotes the first order derivative w.r.t. λ, $\mathsf{H}(v,z)$ is the Hermite function, and ${}_1\mathsf{F}_1(a,b,z)$ is the Kummer confluent hypergeometric function.[4]

(ii) ***Hitting up*** $(0 \le x < y \le 1)$. *$\{\lambda_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are as in (i) with the constant A in (2.9) substituted by $A = -\frac{\theta\sqrt{\kappa}\ {}_1\mathsf{F}_1\left(1-\frac{\lambda}{2\kappa}, \frac{3}{2}, \frac{\theta^2\kappa}{\sigma^2}\right)}{\sigma \mathsf{H}\left(-1+\frac{\lambda}{\kappa}, -\frac{\theta\sqrt{\kappa}}{\sigma}\right)}$.*

Proof of (2.5): Specialize Proposition 2 and Remark 3 in Linetsky [9] to the conditions satisfied by the ROU process.

(i) **Hitting down** $(0 \le y < x \le 1)$. The relevant Sturm-Liouville (SL) problem is

$$-\mathscr{A} f(x) = \lambda f(x). \tag{2.10}$$

on $(y, 1)$, where $\mathscr{A}$ is defined in (2.2). 1 is a regular instantaneously reflecting boundary, and the corresponding boundary condition is the second one in (2.3). The unique (up to a multiple independent of x) solution of the ODE (2.10) with boundary condition $f'(1) = 0$

[4]Both $\mathsf{H}(v,z)$ and ${}_1\mathsf{F}_1(a,b,z)$ are available as build-in functions in MATHEMATICA® with the calls HermiteH$[v,z]$ and Hypergeometric1F1$[a;b;z]$ respectively.

is (2.9), and it satisfies $\int_y^1 |f(x,\lambda)|^2 m(x)\mathrm{d}x < \infty$.[5] Moreover, the spectrum of $\mathscr{A}$ is non-negative, simple and purely discrete (see [11] and [14]).
(ii) **Hitting up** ($0 \le x < y \le 1$). It is similarly treated. □

Proof of (2.6): The distribution of T_y is displayed in (2.5). The only thing to do is to verify the uniformly convergence of (2.6). Here, the asymptotic expressions of ${}_1\mathsf{F}_1(a,b,z)$ when $a \to -\infty$ and $\mathsf{H}(v,z)$ when $v \to \infty$ are needed to derive the estimates of λ_n and c_n. Those are (see p.68 in [15] and p.285 in [8])

$$\begin{aligned} {}_1\mathsf{F}_1(a,b,z) &= \pi^{-1/2}\Gamma(b)e^{z/2}\left(z(b/2-a)\right)^{1/4-b/2} \\ &\quad \times \cos(2\sqrt{z(b/2-a)} - \pi b/2 + \pi/4)\{1+\mathcal{O}(|a|^{-1/2})\}, \end{aligned} \tag{2.11}$$

for fixed $b > 0,\ z > 0$, and

$$\begin{aligned} \mathsf{H}(v,z) &= 2^{v+1/2}e^{z^2/2}(v/2+1/4)^{v/2}e^{-v/2-1/4} \\ &\quad \times \cos(2z\sqrt{v/2+1/4} - v\pi/2)\{1+\mathcal{O}(v^{-1/2})\}. \end{aligned} \tag{2.12}$$

for fixed $z \in \mathbb{R}$. In order to confirm the uniformly convergence of the series in (2.6), we will give some proper estimates of $\{\lambda_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$.
(i) **Hitting down** ($0 \le y < x \le 1$). From (2.7) and (2.9), it follows that $\{\lambda_n\}$ are the roots of the following equation

$$\begin{aligned} &\sigma \mathsf{H}\left(-1+\frac{\lambda}{\kappa}, (1-\theta)\frac{\sqrt{\kappa}}{\sigma}\right) {}_1\mathsf{F}_1\left(-\frac{\lambda}{2\kappa}, \frac{1}{2}, (y-\theta)^2\frac{\kappa}{\sigma^2}\right) \\ = &\ (\theta-1)\sqrt{\kappa}\,{}_1\mathsf{F}_1\left(1-\frac{\lambda}{2\kappa}, \frac{3}{2}, (1-\theta)^2\frac{\kappa}{\sigma^2}\right)\mathsf{H}\left(\frac{\lambda}{\kappa}, (y-\theta)\frac{\sqrt{\kappa}}{\sigma}\right). \end{aligned} \tag{2.13}$$

Using the expressions (2.11), (2.12) and standard calculus, equation (2.13) turns to be

$$\sin\left(\frac{\pi\lambda}{2\kappa}\right)\cos(\alpha-\beta_y)\{1+\mathcal{O}(\lambda^{-\frac{1}{2}})\} = 0, \tag{2.14}$$

where $\alpha = 2(1-\theta)\frac{\sqrt{\kappa}}{\sigma}\sqrt{\frac{\lambda}{2\kappa}-\frac{1}{4}}$ and $\beta_y = 2(y-\theta)\frac{\sqrt{\kappa}}{\sigma}\sqrt{\frac{\lambda}{2\kappa}+\frac{1}{4}}$. From (2.14), when n is large enough, λ_n can be well approximated by one of the elements in the following set

$$\{2\kappa k\}_{k=1}^{\infty} \cup \{s_k\}_{k=1}^{\infty}, \tag{2.15}$$

where s_k is the solutions to the equation $f(\lambda) := \alpha - \beta_y - (\frac{\pi}{2}+k\pi) = 0$. Define $C := \frac{\sigma}{2\sqrt{\kappa}}(\frac{\pi}{2}+k\pi)$. By the standard calculus, we have

$$s_k = \begin{cases} \left(\frac{C(y-\theta)-(1-\theta)\sqrt{C^2-((1-\theta)^2-(y-\theta)^2)/2}}{((1-\theta)^2-(y-\theta)^2)/\sqrt{2\kappa}}\right)^2 + \frac{\kappa}{2} \sim C_1k^2, & \text{if } 2\theta > 1+y, \\ \left(\frac{C(y-\theta)+(1-\theta)\sqrt{C^2-((1-\theta)^2-(y-\theta)^2)/2}}{((1-\theta)^2-(y-\theta)^2)/\sqrt{2\kappa}}\right)^2 + \frac{\kappa}{2} \sim C_2k^2, & \text{otherwise}, \end{cases} \tag{2.16}$$

[5]Here $m(x)$ is the speed density given by $m(x) = \frac{2}{\sigma^2}\exp\left(\frac{\kappa\theta^2}{\sigma^2} - \frac{\kappa(\theta-x)^2}{\sigma^2}\right)$.

where C_1 and C_2 are two positive constants independent of k.

Next we study the large-n approximation for c_n. Recall (2.8). Using the asymptotic expressions (2.11) and (2.12), we have, for large n

$$c_n \approx \frac{-\sin\frac{\pi\lambda_n}{2\kappa}\cos(\alpha-\beta_x)}{\lambda_n\left(-\frac{\pi}{2\kappa}\cos(\frac{\pi\lambda_n}{2\kappa})\cos(\alpha-\beta_y)+\sin\frac{\pi\lambda_n}{2\kappa}\sin(\alpha-\beta_y)g(\lambda_n)\right)},$$

where $g(\lambda_n)=\frac{1-\theta}{2\sigma\sqrt{\kappa}}(\frac{\lambda_n}{2\kappa}-\frac{1}{4})^{-\frac{1}{2}}-\frac{y-\theta}{2\sigma\sqrt{\kappa}}(\frac{\lambda_n}{2\kappa}+\frac{1}{4})^{-\frac{1}{2}}$. Then we have

$$c_n \quad \approx \quad \begin{cases} 0, & \text{if } \lambda_n \approx 2\kappa k, \\ (-1)^{k+1}\frac{\cos(\alpha-\beta_x)}{\lambda_n g(\lambda_n)} \sim \mathcal{O}(\lambda_n^{-\frac{1}{2}}), & \text{if } \lambda_n \approx s_k, \end{cases} \tag{2.17}$$

From (2.15), (2.16) and (2.17), we can conclude that the series in (2.6) is uniformly convergent on $[t_0,\infty)$ for all $t_0>0$.

(ii) **Hitting up** ($0 \le x < y \le 1$). Since the proof is similar, we omit it. □

3. An application to the regulated market

In this section, we use the ROU process on $[0,1]$ to model the price dynamics of the goods or services in a regulated market (see, e.g., [1]). The goal is to compute the conditional default probability (CDP) under the structure framework for credit risk. Here the default is due to down-crossing some threshold level by the underlying price process.

Mathematically, we are concerned with the following quantity

$$F(t;x,y) := P_x(T_y < t), \qquad t>0, \quad 1 \ge x > y \ge 0. \tag{3.1}$$

Here x is the starting point of the process and y is the default barrier. From (2.5), we have $F(t;x,y)=1-\sum_{n=1}^{\infty}c_n e^{-\lambda_n t},\ t>0$, where $\{\lambda_n\}$ and $\{c_n\}$ are given by Proposition (2.1). For simplicity, throughout this section, we adopt the following parameter values: $y=\theta=0.5,\ \kappa=0.25,\ \sigma=0.2$ and $x=0.8$.

n	λ_n	c_n	n	λ_n	c_n	n	λ_n	c_n
2	0.50	0	100	46.0	$-9.50*10^{-11}$	200	95.0	$1.38*10^{-8}$
3	1.00	$-1.48*10^{-17}$	101	46.5	$-1.19*10^{-10}$	201	95.5	$7.05*10^{-9}$
4	1.50	$-1.39*10^{-6}$	102	47.0	$2.21*10^{-10}$	202	96.0	$6.24*10^{-9}$
6	2.00	0	103	47.5	$2.02*10^{-11}$	203	96.5	$5.64*10^{-9}$
7	2.50	$-3.61*10^{-17}$	104	48.0	$-2.59*10^{-10}$	204	97.0	$5.11*10^{-9}$

TABLE 1. *The values of λ_n and c_n when $\lambda_n \approx 2\kappa k, k \in \mathbb{N}^+ := \{1,2,\dots\}$.*

n	λ_n	c_n	n	λ_n	c_n
1	0.34737	1.14216	54	24.0743	−0.12857
5	1.96233	0.0468945	74	33.5492	−0.0332739
13	5.12344	−0.323061	97	44.6032	0.111897
23	9.86158	0.0950119	123	57.2363	−0.03439
37	16.1784	0.141182	152	71.4485	−0.06970

TABLE 2. *The values of λ_n and c_n when $\lambda_n \approx s_k, k \in \mathbb{N}^+$ (recall s_k in (2.16)).*

Table 1 and 2 report the values of $\{\lambda_n\}$ and $\{c_n\}$. These values are consistent with the asymptotic estimates obtained in section 2.

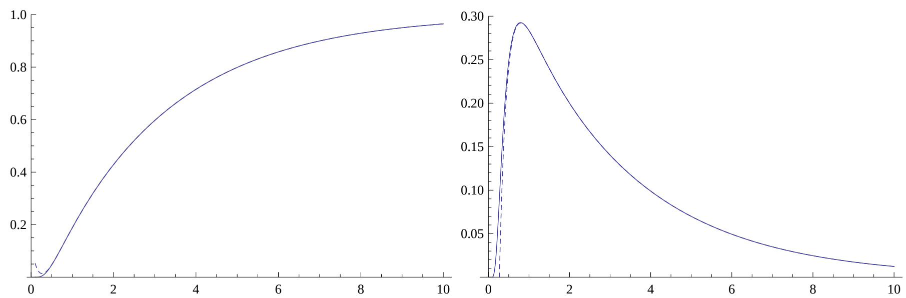

FIG 1. LEFT: *The conditional default probability of the default time. The solid line and the dashed depict the series (2.5) truncated after 251 terms and 20 terms respectively.* RIGHT: *Density functions corresponding to the Left panel.*

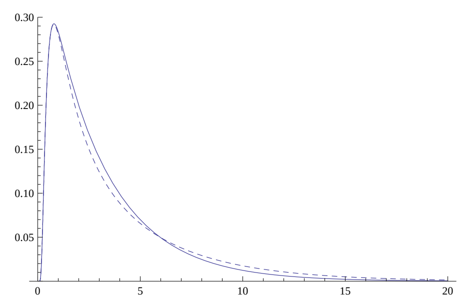

FIG 2. *The densities of the first hitting times for OU process and ROU process with the same parameters. The solid line is for the ROU process (the series (2.5) truncated after 251 terms).*

Figure 1 plots the conditional default probability (i.e., the cumulative distribution function $F(t; 0.8, 0.5)$) and the corresponding density function. Figure 2 compares the density of first hitting time of OU process (see (8) in Göing-Jaeschke and Yor [5]) with the density of first hitting time for ROU process. We find that, due to the existence of the reflecting barrier, the time taken by the ROU process to hit the default barrier y is shorter than that taken by the OU process. This phenomenon is consistent with intuition.

4. Conclusion

In this paper, we have presented the analytic expressions for the densities of first hitting times of reflected Ornstein-Uhlenbeck processes. An application to the regulated market has been included. The numerical results are consistent with both theory and intuition.

References

[1] Bo, L., Tang, D., Wang, Y. and Yang, X. (2010). On the conditional default probability in a regulated market: a structural approach. *Quantitative Finance*, doi:10.1080/14697680903473278.

[2] Bo, L., Wang, Y. and Yang, X., (2010). Some integral functionals of reflected SDEs and their applications in finance. *Quantitative Finance*, doi:10.1080/14697681003785926.

[3] Bo, L., Zhang, L. and Wang, Y. (2006). On the first passage times of reflected O-U processes with two-sided barriers. *Queueing Syst.* **54** 313-316.

[4] Goldstein, R. and Keirstead, W. (1997). On the term structure of interest rates in the presence of reflecting and absorbing boundaries. Preprint.

[5] Göing-Jaeschke, A. and Yor, M. (2003). A clarification about hitting time densities for Ornstein-Uhlenbeck processes. *Finance and Stochastics* **7** 413-415.

[6] Karlin, S. and Taylor, H. M. (1981). *A Second Course in Stochastic Processes.* Academic Press, New York.

[7] Kent, J. (1980) Eigenvalue expansions for diffusion hitting times *Z. Wahrsch. verw. Geb.* **52** 309-319.

[8] Lebedev, N. N. (1972) *Special Functions and Their Applications.* New York: Dover Publications, Inc.

[9] Linetsky, V. (2004a). Lookback options and diffusion hitting times: A spectral expansion approach. *Finance Stochast.* **8** 373-398.

[10] Linetsky, V. (2004b). Computing hitting time densities for CIR and OU diffusions: applications to mean-reverting models. *J. Comput. Finance* **7** 1-22.

[11] Linetsky, V. (2004c). The spectral decomposition of the option value. *Internat. J. Theoret. App. Finance* **7** 337-384.

[12] Linetsky, V. (2005). On the transition densities for reflected diffusions. *Adv. Appl. Prob.* **37** 435-460.

[13] Lions, P. and Sznitman, A. (1984) Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511-537.

[14] McKean, H. (1956). Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.* **82** 519-548.

[15] Slater, L. J. (1960). *Confluent hypergeometric functions.* Cambridge University Press.

[16] Saphores, J. D. M. (2005). The density of bounded diffusions. *Economics Letters* **86** 87-93.

[17] Ward, A. and Glynn, P. (2003a). A diffusion approximation for a Markovian queue with reneging. *Queueing Syst.* **43** 103-128.

[18] Ward, A. and Glynn, P. (2003b). Properties of the reflected Ornstein-Uhlenbeck process. *Queueing Syst.* **44** 109-123.

[19] Ward, A. and Glynn, P. (2005). A diffusion approximation for a GI/GI/1 queue with balking or reneging. *Queueing Syst.* **50** 371-400.