First Passage Time of Reflected Stable Ornstein-Uhlenbeck Processes

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. This space supports a spectrally negative stable process $Z = (Z_t)_{t\geq 0}$ with index $\alpha \in (1, 2]$ and Laplace exponent

$$\Phi(\lambda) = (c\lambda)^\alpha, \qquad \lambda > 0, \tag{1}$$

where $c = \hat{c}|\cos(\pi\alpha/2)|^{-1/\alpha}$ and $\hat{c} > 0$ (see, e.g., Example 46.7 in Sato(1999)).

The Reflected Stable OU Processes

Consider the following stable Ornstein-Uhlenbeck (OU) process with reflection:

$$\mathrm{d}X_t = -\gamma X_t \mathrm{d}t + \mathrm{d}Z_t + \mathrm{d}L_t, \qquad X_0 = x \geq 0, \tag{2}$$

where $\gamma > 0$, and $L = (L_t)_{t \geq 0}$ is the regulator (or local time), which is the magnitude of the minimum displacement that can keep the process $X = (X_t)_{t \geq 0}$ always nonnegative. We have the following properties of L:

(1) L is RCLL, nondecreasing and $L_0 = 0$;

(2) For all $t \geq 0$, $L_t = \int_0^t 1_{\{X_u = 0\}} \mathrm{d}L_u$.

Literature and background

The first passage problems for the generalized OU (GOU) processes without reflections have been investigated by many authors (see, e.g., Hadjiev (1985), Novikov (2004), Patie (2005), Jakubowski (2007) and Borovkov & Novikov (2008)).

Recently the stochastic processes with reflections have attract more and more attention both in theory and in their applications in queueing, insurance and finance.

In this talk, we investigate the first passage problem for the reflected stable OU processes.

First passage time

Define the first passage time for the reflected stable OU process X by

$$T_y = \inf\{t \geq 0;\ X_t \geq y\}, \tag{3}$$

where the hitting level $y \in (x, \infty)$. Since X is spectrally negative, it holds that $X_{T_y} = y$.

We will derive the explicit expression for the Laplace transform of the first passage time T_y.

Main result

Recall $\mathrm{d}X_t = -\gamma X_t \mathrm{d}t + \mathrm{d}Z_t + \mathrm{d}L_t$ and the Laplace exponent of Z, $\Phi(\lambda) = (c\lambda)^\alpha$. We have

Theorem

The Laplace transform of the first passage time T_y is given by

$$\mathbb{E}_x\left[e^{-\theta T_y}\right] = \frac{\mathrm{H}^\alpha_{d\theta}(d^{-1/\alpha}c^{-1}x)}{\mathrm{H}^\alpha_{d\theta}(d^{-1/\alpha}c^{-1}y)}, \qquad 0 \le x < y, \tag{4}$$

where $d = (\alpha\gamma)^{-1}$, and $\mathrm{H}^\alpha_\delta(x) = \sum_{k=0}^\infty \frac{\Gamma(k+\delta)}{\Gamma(\alpha k+1)} x^{\alpha k}$ with $\Gamma(u) = \int_0^\infty e^{-v} v^{u-1} \mathrm{d}v$.

1. Prove that

$$T_y = A^{-1}(\tau_y^d),$$

where $A^{-1}(t) = d \log(t/d + 1)$ with $d := (\alpha\gamma)^{-1}$. Here τ_y^d is a random time defined by

$$\tau_y^d := \inf\{t \geq 0;\ x + \hat{Z}_t - \hat{L}_t \geq y(t/d + 1)^{1/\alpha}\},$$

where $\hat{Z}$ is an α stable process and $\hat{L}_t = \inf_{s \leq t}(x + \hat{Z}_s) \wedge 0$.

Sketch of the proof: step 2 of 3

2. Recall the Laplace transform of the first entrance time τ_z of the reflected α stable process into $[z, \infty)$ (see, e.g., (3.9) in Avram, Palmowski & Pistorius (2007))

$$\mathbb{E}_x \left[\exp\left(-\theta \tau_z\right)\right] = \frac{G(\theta^{1/\alpha} c^{-1} x)}{G(\theta^{1/\alpha} c^{-1} z)}, \qquad \theta > 0,$$

where

$$G(x) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1 + \alpha k)}, \qquad x \in \mathbf{R}.$$

Sketch of the proof: step 3 of 3

3. Complete the proof by noting that the Laplace transform of the first passage time T_y can be rewritten as

$$\mathbb{E}_x \left[\exp\left(-\theta T_y \right) \right] = d^{d\theta} \mathrm{M}(y; x, d, d\theta),$$

and using a martingale related to the Mellin transform of the random time τ_y^d:

$$\mathrm{M}(y; x, d, \delta) := \mathbb{E}_x \left[\left(\tau_y^d + d \right)^{-\delta} \right], \text{ for } \delta > 0.$$

Numerical results

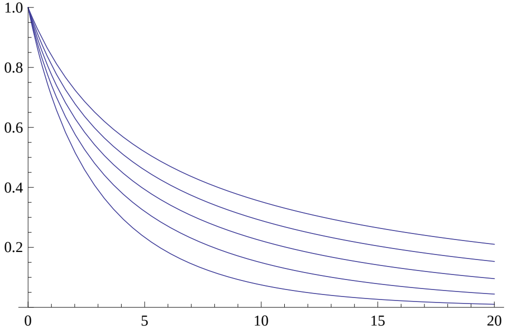

Figure: The Laplace transform with $\alpha = 6/5, 7/5, 8/5, 9/5, 2$ (from the bottom up). The other parameters are $c = \frac{1}{\sqrt{2}}, \gamma = \frac{1}{4}, x = \frac{1}{4}, y = \frac{1}{2}$.

Numerical results (Cont'd)

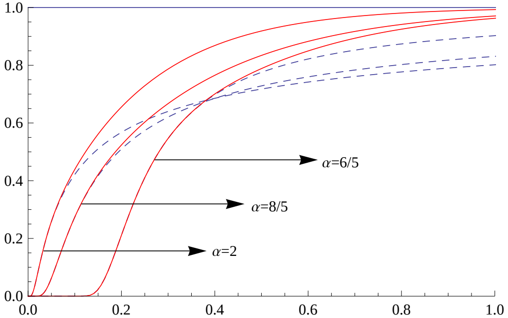

Figure: The density of the first passage time T_y. The dashed lines correspond to stable OU without reflection. The other parameters are $c = 1/\sqrt{2}, \gamma = 1/4, x = 1/4, y = 1/2$.

THANK YOU!