On the Conditional Survival Probability in a Regulated Market: Reflected Ornstein-Uhlenbeck Model

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Outline

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Our Goal

Compute the conditional survival probability in a regulated market under the structural framework using incomplete information.

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Structure Framework for Credit Risk

Consider a firm with market value $(V_t)_{t\geq 0}$. The firm is financed by equity and a zero coupon bond with face value D and maturity date T. Denote the default time by τ, and assume that $V_0 > D$.

Classical approach: Merton (1974)

$$\tau = T, \text{ if } V_T < D; \quad \tau = \infty, \text{ otherwise}$$

First-passage approach: Black & Cox (1976) and this talk

$$\tau = \inf\{t > 0; V_t < D\}$$

Other models: Excursion approach, Reduced-form models.

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Regulated Market and “Why Reflection”

In a regulated market, the goods or services (for instance, grains, water, gas, electricity supply and other important materials or services for a country) are regulated by a government appointed body and the prices are allowed to be charged.

The price control commonly results in the boundedness of the price of these regulated goods or services. This characteristic (boundedness) stimulates us to present a tractable bounded stochastic process to describe the price dynamics of the regulated goods or services.

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Regulated Market and “Why Reflection” (Cont’d)

Krugman (1991): Foreign exchange rate
Goldstein & Keirstead (1997): Interest rate
Veestraeten (2008): Stock price

In this talk, we will use the reflected Ornstein-Uhlenbeck (O-U) process on $[0, b]$ ($b > 0$) to model the price dynamics of the regulated goods or services.

The other bounded processes may also be adopted to formulate the regulated price dynamics.

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Incomplete (Partial) Information

Usually, the complete information on the market price is unavailable. Specifically, we will assume that:

We can only observe the market price at some discrete times, which can be interpreted as the quarterly provided reports on the asset evaluations of the firm (see, e.g., Duffie & Lando (2000)).

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The ROU Model

Let $\Lambda = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with $(\mathcal{G}_t)_{t\geq 0}$ satisfying the usual conditions. $\mathbb{P}$ is the physical (statistical) measure. Suppose that the market price (of some regulated financial variables) follows a *bounded* process $Q = (Q_t)_{t\geq 0}$:

$$\begin{cases} \mathrm{d}Q_t = (\mu - \alpha Q_t)\mathrm{d}t + \sigma \mathrm{d}w_t + \mathrm{d}l_t - \mathrm{d}u_t, \\ Q_0 = v \in [0, b], \end{cases}$$

where $w = (w_t)_{t\geq 0}$ is a standard Brownian motion and $\mu \in \mathbf{R}$, $\alpha, \sigma \in \mathbf{R}^+$. $l = (l_t)_{t\geq 0}$ and $u = (u_t)_{t\geq 0}$ are usually called the regulators of the reflected process Q at points 0 and b.

Krugman (1991) interpreted the regulators as the governmental (or central bank) intervention. (RBM to ROU, mean reversion)

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The ROU Model (Cont'd)

In fact, l and u are the minimum nondecreasing processes that can prevent the process from going outside the band $[0, b]$. They have the following properties (see, e.g., Harrison (1986)):

For $t \in [0, \infty)$, the sample paths $t \to l_t$ and $t \to u_t$ are continuous and $l_0 = u_0 = 0$.

$$\int_0^t \mathbf{1}_{\{Q_s > 0\}} \mathrm{d}l_s = 0, \quad \text{and} \quad \int_0^t \mathbf{1}_{\{Q_s < b\}} \mathrm{d}u_s = 0, \quad \text{for all } t > 0.$$

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The Default Time

Define the default time τ by

$$\tau = \inf\{t \geq 0;\ Q_t \leq d\},$$

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Let $D_t := \mathbf{1}_{\{\tau \leq t\}}$. We call $(D_t)_{t \geq 0}$ the default indicator process.

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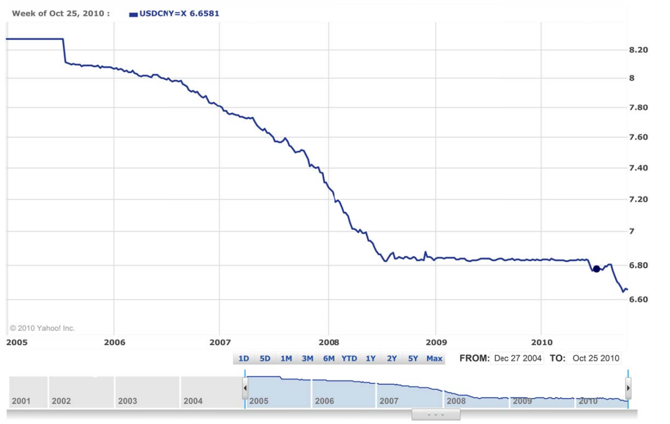
An Example

USD/CNY: From July 21, 2005, floating FX came into effect. Many Chinese firms have been bankrupted due to the appreciation of CNY.



The Incomplete Information

Assume that $0 \leq t_1 < t_2 < \cdots < t_n < \cdots$ are a sequence of deterministic observed times. For each $t > 0$ fixed, define $n_t := \max\{j;\ t_j \leq t\}$.

We denote the observed price at time t_i by $Y_{t_i} := Q_{t_i} + \xi_{t_i}$, where $\xi = (\xi_t)_{t \geq 0}$ is an extra noisy source independent of Q.

The partial information is $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \subset \mathcal{G}$, where

$$\mathcal{F}_t = \sigma\left(\{Y_{t_1}, \cdots, Y_{t_{n_t}}\}\right) \vee \sigma\left(\{D_u;\ 0 \leq u \leq t\}\right).$$

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What Will I Do

For convenience, we introduce the following notation: for every $(t, s) \in [0, \infty) \times [0, \infty)$ with $t > s$, denote the CSP by

$$\ell(s, t, Y_s) := \mathbb{P}(\tau > t | \mathcal{F}_s).$$

We are going to present the explicit expression for the CSP $\ell(s, t, Y_s)$ for the case $t > s$ and $s = t_i$ with $i = 1, 2, \ldots$. We consider the cases of single observation and multiple observations separately.

In this talk, we only depict the result for the case of single observation, i.e., $s = t_1$.

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Some Known Results

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The Laplace transform of the first hitting time of ROU process has been obtained in Bo, Wang and Zhang (2006).

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Theorem

Let $s = t_1$ and $t > s$. Then the Conditional Survival Probability is

$$\ell(s,t,Y_s) = \frac{\int_d^b \mathbb{P}_u(\tau > t-s)h(\mathrm{d}u, Y_s, s)}{\int_d^b h(\mathrm{d}u, Y_s, s)},$$

where

$$\frac{h(\mathrm{d}u, y, s)}{\mathrm{d}u} = \frac{\mathcal{F}_\xi(s; y-\mathrm{d}u)}{\mathcal{F}_Y(s;\mathrm{d}u)}\left[p(s;v,u) - \int_0^s p(s-r;d,u)\mathbb{P}_v(\tau\in\mathrm{d}r)\right]$$

with $p(\cdot;\cdot,\cdot)$ being the transition density of the ROU processes. Here

$$\mathcal{F}_X(t;\mathrm{d}x) := \mathbb{P}_v(X_t\in\mathrm{d}x), \quad t\geq 0.$$

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We next presents some numerical results associated with the CSP.

For parsimony, we will adopt the following preference parameters.

Table: Preference parameters.

drift coefficient μ	0
decay coefficient α	1
spot interest rate $\bar{r}$	0.06
diffusion coefficient σ	1
reflected upper bound b	1
default barrier d	0.25
initial asset value $Q_0 = x_0$	0.5
added noisy source ξ_t	$N(0,1)$

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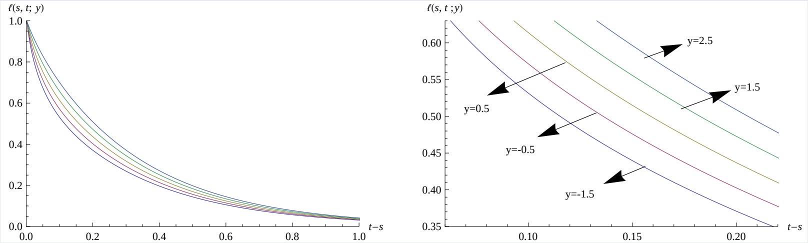

Figure: LEFT: Conditional survival function $\ell(s, t; y)$ for $y = 2.5, 1.5, 0.5, -0.5, -1.5$ with $s = 0.1$; RIGHT: A local display on the axis domain $[0.05, 0.2] \times [0.3, 0.55]$ for the left figure.

Table: Conditional survival probabilities at $X_s = y = -1.5$ and 2.5 with $s = 0.1$. The non-regulated counterparts are given in parenthesis.

	$\ell(s,t;y)$	$y = -1.5$	$y = 2.5$
$t - s =$	0.1	0.532 (0.541)	0.701 (0.698)
	0.2	0.371 (0.386)	0.509 (0.528)
	0.3	0.269 (0.300)	0.372 (0.421)
	0.4	0.198 (0.242)	0.271 (0.345)
	0.5	0.144 (0.201)	0.198 (0.288)
	0.6	0.106 (0.169)	0.145 (0.244)
	0.7	0.078 (0.144)	0.106 (0.208)
	0.8	0.057 (0.123)	0.078 (0.179)
	0.9	0.042 (0.106)	0.057 (0.154)
	1.0	0.031 (0.092)	0.042 (0.134)

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Thank you!